Sultan Qaboos University
Department of Mathematics and Statistics

Math 4141 Fall 2010

Quiz 2 Time: 30 minutes

Name:. . . . . . . Solution. . . . . Section: . . . . . Number. . . . .

In questions 1, 2 and 3, show your complete, mathematically correct and neatly written solution.

Q1: *(4 +1 points)*

(i) Find the Lagrange polynomial that interpolates the points $(0, 1), (1, 2), (2, 0)$.

(ii) True or False. If $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ are $n+1$ distinct points, then the Lagrange interpolating polynomial $P(x)$ of these points is of degree exactly n. If the statement is false, then give a valid justification.

Solution: (i)

$$\begin{aligned} P(x) =& L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) \\ =& L_0(x) + 2L_1(x) + (0) \cdot L_2(x) \\ =& L_0(x) + 2L_1(x) \\ =& \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x2)} + 2\frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x2)} \\ =& \frac{1}{2}(x - 1)(x - 2) - 2x(x - 2) \end{aligned}$$

(ii) False, it is at most n. For example, take the function $f(x) = x + 1$ and take the 4 points $(1, 2), (2, 3), (3, 4), (4, 5)$, then the interpolating polynomial is $P(x) = x+1$, which is of degree 1, not 4.

Q2: Use Newton's interpolatory divided-difference method to find the polynomial that interpolates the points $(0, 1), (1, 2), (2, 3), (3, 0)$. *(4 points)*

Solution:

x	$f(x)$			
0	1			
		1		
1	2		0	
		1		$\frac{-2}{3}$
2	3		-2	
		-3		
3	0			

Hence, the interpolating polynomial is

$$P(x) = 1 + (x - 0) + 0(x - 0)(x - 1) - \frac{2}{3}(x - 0)(x - 1)(x - 2) = 1 + x - \frac{2}{3}x(x - 1)(x - 2).$$

Q3: *(4 +2 points)*

(i) The function $f(x) = \sin(|x|)$ is differentiable everywhere except at $x = 0$. Use the two three-point formulas that we derived with $h = 0.1$ to approximate $f'(0.2)$. Which one of the two formulas is better in this case?

`Solution:` From the first formula, we obtain

$$\begin{aligned} f'(x_0) \approx & \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right] \\ = & \frac{1}{0.2}\left[-3f(0.2) + 4f(0.3) - f(0.4)\right] \\ \approx & 0.983. \end{aligned}$$

From the second formula, we obtain

$$\begin{aligned} f'(x_0) \approx & \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] \\ = & \frac{1}{0.2}\left[f(0.3) - f(0.1)\right] \\ \approx & 0.978. \end{aligned}$$

Since

$$f'(x) = \begin{cases} \cos(x) & x > 0 \\ -\cos(-x) & x < 0, \end{cases}$$

then $f'(0.2) = 0.980$.

In the first formula, the error is $|0.980 - 0.983| = 0.003$.

In the second formula, the error is $|0.978 - 0.980| = 0.002$.

Hence, the second formula gives a better approximation.

(ii) Use the Trapezoidal Rule to approximate $\int_0^1 \cos(x^2)\, dx$.

`Solution:`

$$\int_0^1 \cos(x^2)\, dx \approx \frac{1-0}{2}(cos(0) + cos(1)) \approx 0.77$$

Good Luck