Open Problems*

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Given a continuous function $f : I \to I$, where I is a closed interval (I can be $\mathbb{R}$), the orbit of the autonomous difference equation

$$x_{n+1} = f(x_n) \tag{1}$$

through a point $x_0 \in I$ is defined as

$$\mathcal{O}^+(x_0) := \{x_0, \overbrace{f(x_0)}^{x_1}, \overbrace{f^2(x_0)}^{x_2}, \overbrace{f^3(x_0)}^{x_3}, \ldots\}, \tag{2}$$

where we use the symbol f^n to denote $f \circ \cdots \circ f \circ f$ (n times). We call such orbit an autonomous orbit; it is called periodic of minimal period r (or an r-cycle) if the sequence $x_0, x_1, x_2, \ldots$ is periodic, and r is the smallest positive integer for which $x_{n+r} = x_n$, for all $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$. In 1965, Sharkovsky [3] proved that the existence of an r-cycle of equation (1) assures the existence of k-cycles for all $r \prec k$ in the following ordering.

$$3 \prec 5 \prec \cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec \cdots 2^n \cdot 3 \prec 2^n \cdot 5 \prec \cdots \prec 2^n \prec \cdots \prec 2^2 \prec 2 \prec 1.$$

In periodically fluctuating environments, we consider the the p-periodic difference equation

$$x_{n+1} = f(n \bmod p, x_n) = f_{n \bmod p}(x_n),\ p > 1,\ n \in \mathbb{N}. \tag{3}$$

Here, the p-periodic sequence of functions in Eq. (3)

$$\{f_0, f_1, f_2, \cdots, f_{p-1}\} \subset \mathcal{C}(I), \tag{4}$$

where $\mathcal{C}(I)$ is the space of continuous functions on a closed interval I. Thus, p is the smallest positive integer for which $f_{p+n} = f_p$, $\forall n \in \mathbb{N}$. In this case, we call Eq. (3) a p-periodic difference equation. The nonautonomous orbit of the p-periodic difference equation (3) through a point $x_0 \in I$ is defined as $\mathcal{O}^+(x_0) :=$

*If you are one of my students or former students, you get a symbolic price of \$30 for each problem you solve correctly. If you are not one of my students and you give me the solution for one of these problems, then I owe you a dinner whenever we meet.

$$\{x_0, \overbrace{f_0(x_0)}^{x_1}, \ldots, \overbrace{f_{p-1}\cdots f_0(x_0)}^{x_p}, \overbrace{f_0 f_{p-1}\cdots f_0(x_0)}^{x_{p+1}}, \overbrace{f_1 f_0 f_{p-1}\cdots f_0(x_0)}^{x_{p+2}}, \ldots\}. \quad (5)$$

If r is the smallest positive integer for which $x_{n+r} = x_n, \forall n \in \mathbb{N}$, then the orbit in (5) is called periodic of minimal period r or an r-cycle.

In 2006, AlSharawi et al. [2] extended Sharkovsky's ordering of the positive integers to what they call "p-Sharkovsky's ordering"

$$\begin{array}{l} \mathcal{A}_{p,3} \prec \mathcal{A}_{p,5} \prec \mathcal{A}_{p,7} \prec \ldots \\ \mathcal{A}_{p,2\cdot 3} \prec \mathcal{A}_{p,2\cdot 5} \prec \mathcal{A}_{p,2\cdot 7} \prec \ldots \\ \vdots \\ \mathcal{A}_{p,2^n\cdot 3} \prec \mathcal{A}_{p,2^n\cdot 5} \prec \mathcal{A}_{p,2^n\cdot 7} \prec \ldots \\ \vdots \\ \cdots \prec \mathcal{A}_{p,2^n} \prec \ldots \prec \mathcal{A}_{p,2^2} \prec \mathcal{A}_{p,2} \prec \mathcal{A}_{p,1}, \end{array} \quad (6)$$

where $\mathcal{A}_{p,q} := \{m \in \mathbb{N} : \ lcm(m,p) = pq\}$. Moreover, they proved, among other things, that if the p-periodic difference equation in (3) has a geometric r-cycle, then each set $\mathcal{A}_{p,q}$, $\mathcal{A}_{p,r} \prec \mathcal{A}_{p,q}$, contains at least one period of a geometric cycle. In general, it is unknown which elements of $\mathcal{A}_{p,q}$ are minimal periods and which are not.

Now, let us define what we call the Γ-set of Eq. (3).

Definition 1. The Γ-set of Eq. (3) is the set of positive integers r such that Eq. (3) has a geometric r-cycle and r is not a multiple of p, i.e., $r \notin \{p, 2p, 3p, \ldots\}$.

Open Problem 1

Consider all "well-known" mathematical models of single-species without age structure in a periodically fluctuating environment. Which model has the largest Γ-set?

Open Problem 2

Construct an example where the Γ-set has cardinality $\aleph$ and the maps $f_j,\ 0 \le j \le p-1$ agree on a set of Lebesgue measure zero. `Solved by Ahmad Al-Salman & Ziyad Al-Sharawi.`

Open Problem 3

Consider the set $\mathcal{A}_{p,q}$ and let $r, s \in \mathcal{A}_{p,q}$. Under what conditions does the existence of an r-cycle imply the existence of an s-cycle? `Solved by Ahmad Al-Salman & Ziyad Al-Sharawi.`

References

[1] Z. AlSharawi, Periodic orbits in periodic discrete dynamics, *Computer and Mathematics with Applications.* **56** (2008) 1966–1974.

[2] Z. AlSharawi, J. Angelos, S. Elaydi, An extension of Sharkovsky's theorem to periodic difference equations, *J. Math. Anal. Appl.* **316** (2006) 128–141.

[3] A. N. Sharkovsky, Coexistence of cycles of a continuous transformation of a line into itself, *Ukrain. Math. Zh.* **16** (1964) 61–71 (in Russian).