Open Problems*

Ziyad AlSharawi
Department of Mathematics and Statistics, Sultan Qaboos University
P. O. Box 36, PC 123, Al-Khod, Sultanate of Oman
Email: alsha1zm@squ.edu.om

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The sets $\mathbb{Z}, \mathbb{R}, \mathbb{R}^+, \mathbb{C}, \mathbb{Z}^+, \mathbb{N}$ denote integers, real numbers, positive real numbers, complex numbers, positive integers, and non-negative integers respectively. Let $(\mathbb{X}, |\cdot|)$ stand for a Banach space, then $\mathcal{C}(\mathbb{X} \times \mathbb{N}, \mathbb{X})$ denotes the space of continuous functions on $\mathbb{X}$.

A function $f(x, n) \in \mathcal{C}(\mathbb{X} \times \mathbb{N}, \mathbb{X})$ is said to be almost periodic (AP, for short) in n uniformly for $x \in \mathbb{X}$, if for any $\epsilon > 0$ and compact set K in $\mathbb{X}$, there exists a positive integer $M = M(\epsilon, K)$ such that any set of M consecutive integers $\{N, N+1, ..., N+M\} \subset \mathbb{N}$ contains an integer p for which

$$|f(x, n+p) - f(x, n)| \leq \epsilon, \quad \forall n \in \mathbb{N}.$$

Such a number p is called *an ϵ-translation number* of $f(x, n)$. For simplicity, we call $f(x, n)$ *a discrete AP function,* and we denote the space of such functions by $\mathcal{AP}(\mathbb{X} \times \mathbb{N}, \mathbb{X})$. If f is independent of x, we just use $\mathcal{AP}(\mathbb{N}, \mathbb{X})$ or for short $\mathcal{AP}(\mathbb{N})$. One can use this definition to show that elements of $\mathcal{AP}(\mathbb{N})$ are bounded. Thus, we equip $\mathcal{AP}(\mathbb{N})$ with the supremum norm $||\cdot||_\infty$. The nonautonomous difference equation $x_{n+1} = f(x_n, n),\ n \in \mathbb{N}$, is called *AP difference equation* or *AP discrete process.*

Definition 1. Let $f \in \mathcal{AP}(\mathbb{N})$ and define

$$M_f(\lambda) := \lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k} f(j) e^{-\lambda j i}. \tag{1}$$

The *exponents* of $f(n)$ are the values of λ for which $M_f(\lambda) \neq 0$. $M_f(0)$ is called the *mean value* of f. The set of exponents of f is denoted by $\Lambda(f)$ and called the *Fourier spectrum* of f. *It is not hard to prove that $\Lambda(f)$ is countable.*

Let $\alpha \in \mathbb{C}$, and $f(n), g(n) \in \mathcal{AP}(\mathbb{N})$. Consider the following difference equations:

$$x_{n+1} = \alpha x_n + g(n). \tag{2}$$

$$x_{n+1} = f(n) x_n + g(n). \tag{3}$$

*If you are one of my students or former students, you get a symbolic price of $30 for each problem you solve correctly. If you are not one of my students and you give me the solution for one of these problems, then I owe you a dinner whenever we meet.

Open Problem 1

Suppose $\alpha = e^{i\theta}$, where $\inf_{\lambda \in \Lambda(f)} |\lambda + \theta| = 0$. Can each solution of Eq. (2) be bounded? Does an AP solution exist?

Open Problem 2

Construct an example where $\lim \left(\prod_{j=0}^{n} f(j) \right)^{\frac{1}{n}} = 1$, $F(n) = \prod_{j=0}^{n} f(j)$ is not periodic but rather AP.

Open Problem 3

Can you give an example where each solution of Eq. (3) is bounded but no AP solution exists.

Conjecture 1

If $\Lambda(f) \subseteq \Lambda(g)$, then Eq. (3) has no AP solution. What if $\Lambda(f) \cap \Lambda(g) \neq \emptyset$?

References

[1] Z. AlSharawi and James Angelos, Linear Almost Periodic Difference Equations, Preprint 2007.