Open Problems and Conjectures*

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Population models of single species with nonoverlapping generations and first order feedback are often represented by difference equations of the form

$$x_{n+1} = x_n f(x_n), \qquad n = 0, 1, 2, ... \tag{1}$$

Here, x_n represents the population biomass at generation n, where x_0 is the initial population. In contest competition models, $xf(x)$ is taken to be nondecreasing to reflect an increasing utilization of available resources [5, 4]. We are particularly interested in the problem where $f(x)$ satisfies the following restrictions:

- $xf(x)$ is increasing and bounded.
- $f(0) > 1$.
- $xf(x)$ is differentiable.

When constant yield harvesting is considered on contest competition models of the form in Eq. (1), one has to deal with difference equations of the form

$$x_{n+1} = x_n f(x_n) - h, \quad h > 0 \tag{2}$$

Observe that Eq. (2) has two equilibrium points, say $\bar{x}_{1,h} \leq \bar{x}_{2,h}$ for small values of h, and as h increases $\bar{x}_{1,h}$ increases and $\bar{x}_{2,h}$ decreases till they collide to form a nonhyperbolic equilibrium point at $h = h_{max}$ before they disappear at $h > h_{max}$

Now, define the persistence set $\mathcal{D}_h$ of Eq. (2) as the set

$$\{x_0 \in \mathbb{R}^+ : \quad x_n \geq 0 \quad \text{for all} \quad n > 0\}.$$

Then, it is easy to prove the following [2, 3]:

Theorem 1. *Consider Eq. (2) with $0 < h \leq h_{max}$. Each of the following holds true:*

(i) $\mathcal{D}_h = [\bar{x}_{1,h}, \infty)$

(ii) If $0 < h_1 < h_2 \leq h_{max}$, *then* $\mathcal{D}_{h_1} \supset \mathcal{D}_{h_2}$.

Next, consider models of the from in Eq. (2) but with second order feedback, i.e.,

$$x_{n+1} = x_n f(x_{n-1}) - h, \quad h > 0, \tag{3}$$

where $h > 0$ and $x_0, x_{-1} \in \mathbb{R}^+$. In this case, $\mathcal{D}_h$ is defined as

$$\{(x_{-1}, x_0) \in \mathbb{R}^{+2} : \ x_n > 0 \quad \text{for all} \quad n > 0\}.$$

*If you are one of my students or former students, you get a symbolic price of 30 for each problem you solve correctly. If you are not one of my students and you solve any of these problems, then I owe you a dinner whenever we meet.

A classical example of Eq. (3) is given by Pielou's model with harvesting [1]

$$x_{n+1} = \frac{bx_n}{1 + x_{n-1}} - h, \quad h > 0. \tag{4}$$

Consider $h = 0.90$ and $b = 4.25$, then computer simulations show that $\mathcal{D}_h$ is given by the shaded region in the following graph:

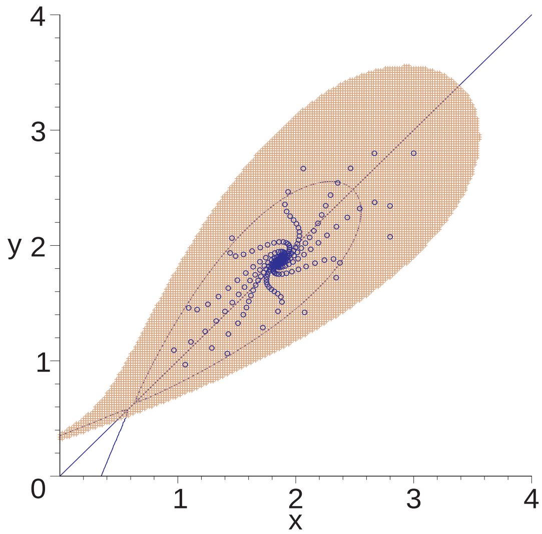

Figure 1: The shaded region in this figure shows the set of initial conditions that survived 500 iterations at $h = 0.90$ and $b = 4.25$, where $x = x_{n-1}$ and $y = x_n$. The blue circles show an orbit oscillating and converging to $\bar{x}_2$. The fading loop shows the largest invariant curve at $h = 1, b = 4.25$.

Conjecture 1: Consider Eq. (4) with $b > 1$. Prove that if $0 < h_1 < h_2$, then $\mathcal{D}_{h_2} \subseteq \mathcal{D}_{h_1}$.

Open Problem 1: Consider Eq. (4) with $b > 4$ and $1 < h < (\sqrt{b} - 1)^2$. What is $\mathcal{D}_h$.

References

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