Sultan Qaboos University
Department of Mathematics and Statistics

Math 4141 Fall 2011

Test 2 Time: 75 minutes

Name:. . . . . .Key. . . . . Section: . . . . . Number. . . . .

Important Instructions

- Make sure you write your name, number and section number on the exam paper and on the solution booklet.
- Solve all questions. Make sure you show your complete, mathematically correct and neatly written solution.
- You are NOT allowed to share calculators or any other material during the test under any circumstances.
- Cellular phones are NOT allowed to be used in class.

Q1: *(3+3+2 points)*

Consider the points $(1,2),(2,1),(3,4),(4,3)$ in answering each of the following:

(i) Use a Lagrange interpolating polynomial of least degree to interpolate the points.

Solution: The Lagrange interpolating polynomial of least degree is obtained by

$$\begin{aligned}P_3(x) =& L_{0,3}(x)f(x_0) + L_{1,3}(x)f(x_1) + L_{2,3}(x)f(x_2) + L_{2,3}(x)f(x_3)\\ =& 2L_{0,3}(x) + L_{1,3}(x) + 4L_{2,3}(x) + 3L_{2,3}(x)\\ =& \frac{-1}{3}(x-2)(x-3)(x-4) + \frac{1}{2}(x-1)(x-3)(x-4)\\ & -2(x-1)(x-2)(x-4) + \frac{1}{2}(x-1)(x-2)(x-3).\end{aligned}$$

(ii) Use the divided differences method to interpolate the points.

Solution:

x	$f(x)$			
1	2			
		-1		
2	1		2	
		3		$\frac{-4}{3}$
3	4		-2	
		-1		
4	3			

Hence, the interpolating polynomial is

$$P_3(x) = 2 - (x-1) + 2(x-1)(x-2) - \frac{4}{3}(x-1)(x-2).$$

(iii) Is the polynomial you obtain in (i) same as the one you obtain in (ii). Justify your answer.

`Solution: Yes, they must be the same because the interpolating polynomial of least degree is unique.`

Q2: Answer each of the following: *(4+2+3 points)*

(i) Use a suitable three-point formula (make sure you write the formula you use) to determine each missing entry in the following table:

x	$f(x)$	$f'(x)$
8.1	16.944	
8.3	17.565	
8.5	18.191	3.140
8.7	18.820	3.164

Solution: The three-point formulas are given by

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right] + \frac{h^2}{3}f'''(\xi), \quad x_0 < \xi < x_0 + 2h$$

and

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] - \frac{h^2}{3}f'''(\xi), \quad x_0 - h < \xi < x_0 + h.$$

We can use the first formula to approximate both $f'(8.1)$ and $f'(8.3)$. Indeed,

$$f'(8.1) \approx \frac{1}{0.4}\left[-3f(8.1) + 4f(8.3) - f(8.5)\right] = 3.0925$$

and

$$f'(8.3) \approx \frac{1}{0.4}\left[-3f(8.3) + 4f(8.5) - f(8.7)\right] = 3.1225.$$

(ii) If you know that the data in the above table is for $y = x\ln(x)$. What is the actual error in each case?

Solution: The actual error is | exact-approximation |. Since $f'(x) = \ln(x) + 1$, then the actual error at 8.1 is

$$|f'(8.1) - 3.0925| = |\ln(8.1) + 1 - 3.0925| = 0.635938\ 10^{-3},$$

and the actual error at 8.3 is

$$|f'(8.3) - 3.1225| = |\ln(8.3) + 1 - 3.1225| = 0.6244485\ 10^{-2}.$$

(iii) Use the error formula to find an error bound when you approximate $f'(8.1)$.

Solution: From the first formula, the error is given by

$$\frac{0.2^2}{3}f'''(\xi), \quad 8.1 < \xi < 8.5.$$

Since, $f'''(\xi) = \frac{-1}{\xi^2}$, which is decreasing in ξ, then

$$|E(8.1)| = |\frac{0.2^2}{3}f'''(\xi)| = \frac{0.2^2}{3}\frac{1}{\xi^2} \leq \frac{0.2^2}{3}\frac{1}{8.1^2} = 0.0002.$$

Q3: *(6 points)*

Let $h = \frac{1}{3}(b-a), x_0 = a, x_1 = a+h$ and $x_2 = b$. Find the degree of precision of the quadrature formula

$$\int_a^b f(x)dx = \frac{9}{4}hf(x_1) + \frac{3}{4}hf(x_2).$$

Solution: The degree of precision of a quadrature formula is the largest positive integer n such that the formula is exact for $x^k,\ k = 0, 1, 2, \ldots, n$. So, we have to take $f(x) = 1, x, x^2, \ldots$ and see when we obtain the formula exact and when it is not exact. Now, since h, x_0, x_1 and x_2 are given, we can write the formula as

$$\int_a^b f(x)dx = \frac{3}{4}h\left(3f(x_1) + f(x_2)\right) = \frac{b-a}{4}\left(3f(\frac{1}{3}(2a+b)) + f(b)\right).$$

For $f(x) = 1$, we obtain

$$L.H.S. = \int_a^b dx = b - a$$

and

$$R.H.S. = \frac{b-a}{4}(3+1) = b - a.$$

So, the formula is exact.

For $f(x) = x$, we obtain

$$L.H.S. = \int_a^b xdx = \frac{1}{2}(b^2 - a^2)$$

and

$$R.H.S. = \frac{b-a}{4}\left(3 \cdot \frac{1}{3}(2a+b) + b\right) = \frac{1}{2}(b^2 - a^2).$$

So, the formula is exact.

For $f(x) = x^2$, we obtain

$$L.H.S. = \int_a^b x^2dx = \frac{1}{3}(b^3 - a^3)$$

and

$$R.H.S. = \frac{b-a}{4}\left(3 \cdot \frac{1}{3^2}(2a+b)^2 + b^2\right) = \frac{1}{3}(b^3 - a^3).$$

So, the formula is exact.

For $f(x) = x^3$, we obtain

$$L.H.S. = \int_a^b x^3dx = \frac{1}{4}(b^4 - a^4)$$

and

$$R.H.S. = \frac{b-a}{4}\left(3 \cdot \frac{1}{3^3}(2a+b)^3 + b^3\right) \neq \frac{1}{4}(b^4 - a^4).$$

Hence, the degree of precision is 2.

Q4: *(6 points)*

Approximate the integral $\int_0^1 e^{-x^2}dx$ using

(i) the Trapezoidal rule (ii) Simpson's rule.

`Solution: (i)`

$$\int_0^1 e^{-x^2}dx = \frac{h}{2}(f(0)+f(1)) - \frac{h^3}{12}f''(\xi)$$
$$\approx \frac{1}{2}(1+e^{-1})$$
$$\approx 0.684$$

`Solution: (ii)`

$$\int_0^1 e^{-x^2}dx = \frac{h}{3}(f(0)+4f(\frac{1}{2})+f(1)) - \frac{h^5}{90}f^{(4)}(\xi)$$
$$\approx \frac{1}{6}(1+4e^{-0.24}+e^{-1})$$
$$\approx 0.747$$

Q5: *(4+3+3+1 points)*

Consider the nonlinear system

$$x_1^2 - 10x_1 + x_2^2 + 8 = 0$$
$$x_1x_2^2 + x_1 - 10x_2 + 8 = 0.$$

and the set

$$D = \{[x_1, x_2]^t : \ 0 \le x_1 \le 1.5, \ 0 \le x_2 \le 1.5\},$$

then answer each of the following:

(i) Write the system as $x_1 = g_1(x_1, x_2)$ and $x_2 = g_2(x_1, x_2)$ such that $G = [g_1, g_2]^t$ maps the set D into itself.

`Solution:`

`From the first equation, we can write` $x_1 = \frac{1}{10}(x_1^2 + x_2^2 + 8)$.

`From the second equation, we can write` $x_2 = \frac{1}{10}(x_1x_2^2 + x_1 + 8)$.

`Now, because`

$$0 \le |g_1(x_1, x_2)| = |\frac{1}{10}(x_1^2 + x_2^2 + 8)| \le \frac{1}{10}(1.5^2 + 1.5^2 + 8) = 1.25 < 1.5$$

`and`

$$0 \le |g_2(x_1, x_2)| = |\frac{1}{10}(x_1x_2^2 + x_1 + 8)| \le \frac{1}{10}(1.5^3 + 1.5 + 8) = 1.2875 < 1.5$$

`then` G `maps` D `into itself.`

(ii) Show that for some $k < 1$, we have

$$\left|\frac{\partial g_i(x_1, x_2)}{\partial x_j}\right| \leq \frac{k}{2}$$

for each $j = 1, 2$ and each $i = 1, 2$.

`Solution: We have`

$$\left|\frac{\partial g_1(x_1, x_2)}{\partial x_1}\right| = \frac{1}{5}|x_1| \leq \frac{3}{10}.$$

$$\left|\frac{\partial g_1(x_1, x_2)}{\partial x_2}\right| = \frac{1}{5}|x_2| \leq \frac{3}{10}.$$

$$\left|\frac{\partial g_2(x_1, x_2)}{\partial x_1}\right| = \frac{1}{10}|x_2^2 + 1| \leq \frac{1.5^2 + 1}{10} = 0.325.$$

$$\left|\frac{\partial g_2(x_1, x_2)}{\partial x_2}\right| = \frac{1}{5}|x_1 x_2| \leq \frac{1.5^2}{10} = 0.45.$$

`Hence,`

$$\left|\frac{\partial g_i(x_1, x_2)}{\partial x_j}\right| \leq 0.45,$$

`and therefore, we can take` $k = 0.9$.

(iii) Start with $X_0 = [0, 1]^t$ and use $X_{k+1} = G(X_k)$ to find X_1 and X_2.

`Solution:`

$$X_1 = G(X_0) = G\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} g_1(0, 1) \\ g_2(0, 1) \end{bmatrix} = \begin{bmatrix} \frac{9}{10} \\ \frac{8}{10} \end{bmatrix}.$$

`and`

$$X_2 = G(X_1) = G\left(\begin{bmatrix} \frac{9}{10} \\ \frac{8}{10} \end{bmatrix}\right) = \begin{bmatrix} g_1(\frac{9}{10}, \frac{8}{10}) \\ g_2(\frac{9}{10}, \frac{8}{10}) \end{bmatrix} \approx \begin{bmatrix} 0.945 \\ 0.9476 \end{bmatrix}.$$

(iv) Evaluate $||X_1 - X_0||_\infty$.

`Solution:`

$$||X_1 - X_0||_\infty = ||[0.9, 0.8]^t - [0, 1]^t||_\infty = ||[0.9, -0.2]^t||_\infty = 0.9.$$

Good Luck