Sultan Qaboos University

Department of Mathematics and Statistics

Math 4141 Fall 2011

Test I Time: 70 minutes

Name:. . . . . Key. . . . . Section: . . . . . Number. . . . .

Important Instructions

- Make sure you write your name, number and section number on the exam paper and on the solution booklet.
- Solve all questions (**Q1** through **Q7**). Make sure you show your complete, mathematically correct and neatly written solution.
- You are NOT allowed to share calculators or any other material during the test under any circumstances.
- Cellular phones are NOT allowed to be used in class.

Q1: Write the correct definition of each of the following: *(2+2 points)*

(i) A sequence $\{q_n\}$ is linearly convergent to q.

`Solution: See Definition 2.6, page 78.`

(ii) A function $f(x)$ has a zero of multiplicity 5.

`Solution: See Definition 2.9, page 82.`

Q2: Write correct and complete statement for each of the following: *(3+3 points)*

(i) Fundamental Theorem of Algebra.

`Solution: See Theorem 2.15, page 91.`

(ii) Fixed Point Theorem.

`Solution: See Theorem 2.3, page 61.`

Q3: Let $f(x) = (x+2)(x+1)^2(x-1)^3(x-2)$. To which zero of f does the Bisection method converge when applied on the interval $[-3, 0]$. Justify your answer. *(4 points)*

Solution: Observe that f has two zeros in the interval $[-3, 0]$, namely $x = -2, -1$. However, when we use the Bisection method, we find f(-3)<0, f(0)>0, $f(-1.5) > 0$. Thus, we take $[-3, -1.5]$ as the new interval. Now, this new interval contains only one zero, and it is $x = -2$. Hence, the Bisection method must converge to -2.

Q4: Let $f(x) = 1 - 5\dfrac{\ln(x)}{x}$. Answer each of the following: *(2+3 points)*

(i) Show that $f(x)$ has a zero in the interval $[1, 20]$.

Solution: We have $f(1) = 1$ and $f(20) \approx 0.251$; however, these points do not help us to apply the I.V.T. So, we test f at a point inside the interval. For instance $f(10) \approx -0.151$. Now, since $f(x)$ is continuous on the interval $[10, 20]$ and $f(10)f(20) < 0$, then by the I.V.T., f has at least one zero in the interval $[10, 20] \subset [1, 20]$. Hence, f has a zero in the interval $[1, 20]$.

(ii) Use Newton's method with $p_0 = 20$ and find p_1, p_2.

`Solution: We have` $f'(x) = \frac{5(-1+\ln(x))}{x^2}$. Therefore,

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 20 - \frac{(1-(1/4)\ln(20))}{(-1/80+(1/80)\ln(20))} \approx 9.9358.$$

and

$$p_2 = 9.9358 - \frac{f(9.9358)}{f'(9.9358)} \approx 12.304.$$

Q5: Let $g(x) = \frac{1}{2}(x + \frac{2}{x})$. Find an interval $[a, b]$ such that $g : [a, b] \to [a, b]$. Justify your answer. *(4 points)*

`Solution: We take` $a = 1, b = 2$ `and show that` $g : [1, 2] \to [1, 2]$. `Since`

$$g'(x) = \frac{1}{2} - \frac{1}{x^2} = \frac{2x^2}{x^2-2},$$

`we obtain` $g'(x) = 0$ `at` $x = \sqrt{2}$. `Thus, the extreme values are`

$$g(1) = \frac{3}{2}, \quad g(\sqrt{2}) = \sqrt{2}, \quad g(2) = \frac{3}{2}.$$

`Since the extreme values are in the interval` $[1, 2]$, `we obtain` $g : [1, 2] \to [1, 2]$.

Q6: Show that $f(x) = \sin(x) - x$ has a zero of multiplicity three at $x = 0$. *(3 points)*

`Solution: We use Theorem 2.11. Since`

$$f'(x) = \cos(x) - 1, \qquad f''(x) = -\sin(x), \qquad f'''(x) = -\cos(x)$$

`and`

$$f(0) = 0, \quad f'(0) = 0, \qquad f''(0) = 0, \qquad f'''(0) = -1 \neq 0.$$

`then` f `has a zero of multiplicity three at` $x = 0$.

Q7: Consider the function $g(x) = 1 + \sin^2(x)$, and let $p_0^{(0)} = 1$. Use Steffensen's method to find $p_0^{(1)}$ and $p_0^{(2)}$. *(4 points)*

`Solution: We use`

$$p_{j+1}^{(k)} = g(p_j^{(k)}), \text{ for } j = 0, 1, \quad \text{and} \quad p_0^{(k+1)} = p_0^{(k)} - \frac{(p_1^{(k)} - p_0^{(k)})^2}{p_2^{(k)} - 2p_1^{(k)} + p_0^{(k)}},$$

`then summarize the computations in the following table:`

k	$p_0^{(k)}$	$p_1^{(k)}$	$p_2^{(k)}$
0	1	1.708	1.981
1	2.153	1.698	1.984
2	1.873		

`Hence` $p_0^{(1)} \approx 2.153$ `and` $p_0^{(2)} \approx 1.873$.

`Total: 30 points`

Good Luck