SULTAN QABOOS UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS

Math 4141 — Fall 2011

Final Exam — Time: 2 hours & 30 minutes

Name:. . . . . . . Key . . . .   Section: . . . . .   Number. . . . .

Important Instructions

- Make sure you write your name, number and section number on the exam paper and on the solution booklet.
- Solve all questions, and make sure you show your complete, mathematically correct and neatly written solution.
- You are NOT allowed to share calculators or any other material during the test under any circumstances.
- Cellular phones are NOT allowed to be used in class.

Total Points *(80 points)*

Q1: *(3+3 points)*

(i) What does it mean when we say "a function $f(x)$ has a zero of multiplicity m at $x = p$"?

`Solution:` A function $f(x)$ has a zero of multiplicity m at $x = p$ if we can write

$$f(x) = (x - p)^m q(x)$$

for some function $q(x)$ that satisfies

$$\lim_{x \to p} q(x) \neq 0.$$

(ii) Show that $f(x) = \cos(x) - 1$ has a zero of multiplicity 2 at $x = 0$.

`Solution:` We take

$$f(x) = x^2 \frac{\cos(x) - 1}{x^2}.$$

Since

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x^2} = \lim_{x \to 0} \frac{-\sin(x)}{2x} = \lim_{x \to 0} \frac{-\cos(x)}{2} = \frac{-1}{2},$$

we obtain that $f(x) = \cos(x) - 1$ has a zero of multiplicity 2 at $x = 0$.

Q2: *(2+3 points)*

(i) What does it mean when we say "a function $f(x)$ has a fixed point at $x = p$"?

`Solution:` A function $f(x)$ has a fixed point at $x = p$ if $f(p) = p$.

(ii) Give an example of a function that has two fixed points. Justify your answer.

`Solution:` We give the following simple example

$$g(x) = x(3 - x).$$

We have two fixed points, namely $x = 0$ and $x = 2$.

Q3: *(4+4+3 points)*

(i) Show that $g(x) = \frac{1}{2}(10 - x^3)^{\frac{1}{2}}$ maps the interval $[1, 1.5]$ into itself.

`Solution:` First, we show that $g(x)$ is decreasing. Because

$$g'(x) = \frac{-3x^2}{4\sqrt{10 - x^3}} < 0 \quad \text{for all} \quad x \in [1, 1.5],$$

we find that g is decreasing. Now, we test the extreme values which are at the end points. At $x = 1$, we have

$$g(1) = \frac{3}{2}.$$

Also, at $x = 1.5$, we have $g(1.5) \approx 1.29$. Hence, the function maps the closed interval $[1, 1.5]$ into itself.

(ii) Prove that g has a unique fixed point in the interval $[1, 1.5]$.

`Solution:` From part (i), there must be a fixed point. Suppose we have two fixed points, Say x_1 and x_2 such that $x_1 < x_2$. Now

$$x_1 - x_2 = g(x_1) - g(x_2) < 0$$

implies that $g(x_1) < g(x_2)$, which is not possible because we have shown in Part (i) that g is decreasing.

(iii) Start with $p_0 = 1$ and find p_1, p_2 and p_3 in the iteration of $p_{n+1} = g(p_n)$.

`Solution:`

$$\begin{aligned} p_1 =& g(p_0) = g(1) = 1.5 \\ p_2 =& g(p_1) = g(1.5) \approx 1.29 \\ p_3 =& g(p_2) = g(1.29) \approx 1.40. \end{aligned}$$

Q4: *(3+4 points)*

(i) Show that $f(x) = \frac{1}{2}(10 - x^3)^{\frac{1}{2}} - x$ has a zero in the interval $[1, 1.5]$.

`Solution:` A zero of $f(x)$ in the interval $[1, 1.5]$ is a fixed point of $g(x)$ in Q3, which we have already verified. Hence, $f(x)$ has a zero in the interval $[1, 1.5]$. Alternatively, one can solve this question by showing that $f(1)f(1.5) < 0$, and use the fact that f is continuous.

(ii) Start with $p_0 = 1$ and use Newton's method to find p_1, p_2 and p_3.

`Solution:` Newton's method is given by

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = p_n - \frac{\frac{1}{2}\sqrt{10 - p_n^3} - p_n}{\frac{-3p_n^2}{4\sqrt{10 - p_n^3}} - 1} = \frac{p_n^3 + 20}{3p_n^2 + 4\sqrt{10 - p_n^3}}.$$

Thus,

$$\begin{aligned} p_1 =& \frac{p_0^3 + 20}{3p_0^2 + 4\sqrt{10 - p_0^3}} \\ =& \frac{1 + 20}{3 + 4\sqrt{10 - 1}} \\ =& 1.4 \end{aligned}$$

$$
\begin{aligned}
p_2 =& \frac{p_1^3 + 20}{3p_1^2 + 4\sqrt{10 - p_1^3}} \\
=& \frac{(1.4)^3 + 20}{3(1.4)^2 + 4\sqrt{10 - (1.4)^3}} \\
\approx& 1.3656
\end{aligned}
$$

$$
\begin{aligned}
p_3 =& \frac{p_2^3 + 20}{3p_2^2 + 4\sqrt{10 - p_2^3}} \\
=& \frac{(1.3656)^3 + 20}{3(1.3656)^2 + 4\sqrt{10 - (1.3656)^3}} \\
\approx& 1.3652.
\end{aligned}
$$

Q5: *(4+5 points)*

(i) Let $a \neq b$. Interpolate the points $(a, f(a)), (b, f(b))$ with a linear polynomial $P_1(x)$.

`Solution:` Since the interpolating polynomial is linear, then we can find it straightforward by finding the equation of the line connecting the two points. Indeed $y = mx + c$, where

$$
m = \frac{f(a) - f(b)}{a - b} \quad \text{and} \quad c = \frac{af(b) - bf(a)}{a - b}.
$$

(ii) Take $P_1(x)$ to be the linear polynomial obtained in part (i), and show that $\int_a^b P_1(x)\, dx$ gives the approximation given in the Trapezoidal rule.

`Solution:`

$$
\begin{aligned}
\int_a^b P_1(x)\, dx =& \int_a^b \left(\frac{f(a) - f(b)}{a - b} x + \frac{af(b) - bf(a)}{a - b} \right) dx \\
=& \frac{f(a) - f(b)}{a - b} \left(\frac{b^2}{2} - \frac{a^2}{2} \right) + \frac{af(b) - bf(a)}{a - b} (b - a) \\
=& \frac{b - a}{2} \left(f(a) + f(b) \right) \\
=& \frac{h}{2} \left(f(a) + f(b) \right), \quad \text{where} \quad h = (b - a).
\end{aligned}
$$

Q6: *(4+5 points)*

(i) Use Simpson's rule to approximate $\int_0^1 \tan(x^2)\, dx$

`Solution:`

$$
\begin{aligned}
\int_0^1 \tan(x^2)\, dx \approx& \frac{1}{6} \left(\tan(0) + 4\tan(0.25) + \tan(1) \right) \\
\approx& 0.43.
\end{aligned}
$$

(ii) Use the composite Trapezoidal rule with $n = 5$ to approximate $\int_0^1 \tan(x^2)\, dx$.

`Solution:` When $n = 5$, we have $h = 0.2$. Thus,

$$\begin{aligned}\int_0^1 \tan(x^2)\, dx \approx& \frac{1}{10}\left(\tan(0) + 2\tan(0.2^2) + 2\tan(0.4^2) + 2\tan(0.6^2) + 2\tan(0.8^2) + \tan(1)\right)\\ \approx& 0.42.\end{aligned}$$

Q7: Find the constants x_0, x_1 and c_1 so that the quadrature formula *(6 points)*

$$\int_0^1 f(x)\, dx = \frac{1}{2} f(x_0) + c_1 f(x_1)$$

has the highest possible degree of precision.

`Solution:` First, take $f(x) = 1$, we obtain

$$\int_0^1 1\, dx = 1 = \frac{1}{2} + c_1.$$

Thus, $c_1 = \frac{1}{2}$ and consequently the formula becomes

$$\int_0^1 f(x)\, dx = \frac{1}{2}\left(f(x_0) + f(x_1)\right).$$

Now, we take $f(x) = x$ to obtain

$$\int_0^1 x\, dx = \frac{1}{2} = \frac{1}{2}\left(x_0 + x_1\right),$$

or equivalently

$$1 - x_0 = x_1.$$

Next, we take $f(x) = x^2$ to obtain

$$\int_0^1 x^2\, dx = \frac{1}{3} = \frac{1}{2}\left(x_0^2 + (1 - x_0)^2\right),$$

or equivalently

$$\frac{2}{3} = 2x_0^2 - 2x_0 + 1.$$

Solve the equation to obtain

$$x_0 = \frac{1}{2} \pm \frac{1}{6}\sqrt{3}.$$

However, since $x_0 + x_1 = 1$, we can take

$$x_0 = \frac{1}{2} - \frac{1}{6}\sqrt{3} \quad \text{and} \quad x_1 = \frac{1}{2} + \frac{1}{6}\sqrt{3}.$$

Now, the quadrature formula becomes

$$\int_0^1 f(x)\, dx = \frac{1}{2}\left(f\left(\frac{1}{2} - \frac{1}{6}\sqrt{3}\right) + f\left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right)\right).$$

Next, we test the formula for $f(x) = x^3$ to find

$$\int_0^1 x^3 \, dx = \frac{1}{4}$$

and

$$\frac{1}{2}\left(\left(\frac{1}{2}-\frac{1}{6}\sqrt{3}\right)^3+\left(\frac{1}{2}+\frac{1}{6}\sqrt{3}\right)^3\right)=\frac{1}{4}.$$

Finally, we test the formula for $f(x) = x^4$ to find

$$\int_0^1 x^4 \, dx = \frac{1}{5}$$

and

$$\frac{1}{2}\left(\left(\frac{1}{2}-\frac{1}{6}\sqrt{3}\right)^4+\left(\frac{1}{2}+\frac{1}{6}\sqrt{3}\right)^4\right)=\frac{7}{36}.$$

Hence, the highest degree of precision is 3.

Q8: Given the initial value problem *(2+5+5 points)*

$$y' = \frac{2}{t}y + t^2 e^t, \quad 1 \le t \le 2, \; y(1) = 0.$$

Answer each of the following:

(i) Show that $f(t,y) = \frac{2}{t}y + t^2e^t$ satisfies the Lipschitz condition.
`Solution:` We have

$$|f(t,y) - f(t,x)| = \frac{2}{|t|}|y-x| \le 2|y-x|.$$

Hence, $f(t,y)$ satisfies the Lipschits condition in y.

(ii) Use Taylor's method of order two with $h = 0.25$ to approximate the solution.
`Solution:` We have $t_0 = 1, t_1 = 1.25, t_2 = 1.5, t_3 = 1.75, t_4 = 2.0$ and

$$\begin{aligned} w_0 &= 0 \\ w_{k+1} &= w_k + h(f(t_k, w_k) + \frac{h}{2}f'(t_k, w_k)) \end{aligned}$$

Because

$$\begin{aligned} f'(t,y) &= \frac{2}{t}y' + \frac{-2}{t^2}y + t^2e^t + 2te^t \\ &= \frac{2}{t}\left(\frac{2}{t}y + t^2e^t\right) + \frac{-2}{t^2}y + te^t(t+2) \\ &= \frac{2}{t^2}y + te^t(t+4), \end{aligned}$$

we obtain

$$\begin{aligned} w_0 &= 0 \\ w_{k+1} &= w_k + h\left(\frac{2}{t_k^2}w_k + t_ke^{t_k}(t_k+4)\right). \end{aligned}$$

Now, we can evaluate to obtain

$$
\begin{aligned}
w_0 &= 0 \\
w_1 &\approx 0 + 0.25\left(0 + e^1(1+4)\right) = \frac{5}{4}e \approx 3.4 \\
w_2 &\approx 3.4 + 0.25\left(\frac{2}{(1.25)^2}(3.4) + (1.25)e^{1.2}(1.25+4)\right) \approx 10.2 \\
w_3 &\approx (10.2) + 0.25\left(\frac{2}{(1.5)^2}(10.2) + (1.5)e^{1.4}(1.5+4)\right) \approx 21.7 \\
w_4 &\approx (21.7) + 0.25\left(\frac{2}{(1.75)^2}(21.7) + (1.75)e^{1.6}(1.75+4)\right) \approx 39.7
\end{aligned}
$$

(iii) Use a Runge-Kutta Method (Midpoint Method) with $h = 0.25$ to approximate the solution of the initial value problem.

`Solution:` Again, we have $t_0 = 1, t_1 = 1.25, t_2 = 1.5, t_3 = 1.75, t_4 = 2.0$ and

$$
\begin{aligned}
w_0 &= 0 \\
w_{k+1} &= w_k + (0.25) f\left(t_k + \frac{0.25}{2}, w_k + \frac{0.25}{2} f(t_k, w_k)\right),
\end{aligned}
$$

where $f(t,w) = \frac{2}{t}w + t^2 e^t$. Now, we do the computation to obtain

$$
\begin{aligned}
w_0 &= 0 \\
w_1 &= 0 + (0.25) f\left(1 + \frac{0.25}{2}, 0 + \frac{0.25}{2} f(1,0)\right) \approx 1.13 \\
w_2 &= 1.13 + (0.25) f\left(1.25 + \frac{0.25}{2}, 1.13 + \frac{0.25}{2} f(1.25, 1.13)\right) \approx 3.73 \\
w_3 &= 3.73 + (0.25) f\left(1.5 + \frac{0.25}{2}, 3.73 + \frac{0.25}{2} f(1.5, 3.73)\right) \approx 8.8 \\
w_4 &= 8.8 + (0.25) f\left(1.75 + \frac{0.25}{2}, 8.8 + \frac{0.25}{2} f(1.75, 8.8)\right) \approx 17.8
\end{aligned}
$$

Q9: Given the matrix *(3+2 points)*

$$
A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 5 & 1 \\ -1 & -1 & 9 \end{bmatrix}.
$$

(i) Use the Gerschgorin theorem (Gerschgorin Circle) to determine the location of the eigenvalues.

`Solution:` We find the three disks centered at $(1,0), (5,0), (9,0)$ respectively

$$
\begin{aligned}
D_1 &= \{(x,y) : \quad (x-1)^2 + y^2 \le 4\} \\
D_2 &= \{(x,y) : \quad (x-5)^2 + y^2 \le 4\} \\
D_3 &= \{(x,y) : \quad (x-9)^2 + y^2 \le 4\}
\end{aligned}
$$

Hence, the eigenvalues are contained in the union of the three disks.

(ii) Depending on (i), how many of the eigenvalues are going to be real numbers? Justify your answer.

`Solution:` All eigenvalues must be real because the three disks obtain in Part (i) are disjoint, and therefore, each one of the disks contains exactly one eigenvalue. Notice that if a disk contains a non-real eigenvalue, then it must contain its conjugate eigenvalue.

Q10: `(True or False questions)` Write `True` beside the true statement and `False` beside the false one in each of the following: *(1 point each)*

(1) The equation $x\cos(x) - 2x^2 + 3x - 1 = 0$ has at least one solution in the interval $[0, 1]$.

(2) The convergence in the fixed point iteration $p_{n+1} = g(p_n)$ is always linear convergence.

(3) The convergence in Newton's method is linear convergence.

(4) Aitken's Δ^2 method can be used to accelerate the convergence of a linearly convergent sequence.

(5) A polynomial of degree n has n real roots.

(6) If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct numbers and f is a function whose values are given at those numbers, then there exists a unique polynomial $P(x)$ of degree at most n such that $f(x_k) = P(x_k)$ for all $k = 0, 1, ..., n$.

(7) Simpson's rule is accurate for quadratic polynomials.

(8) The triangle with vertices $(-2, 0), (0, 3), (2, 0)$ forms a convex set in $\mathbb{R}^2$.

(9) If $V_1 = (1, 2, 3)$ and $V_2 = (3, 2, 1)$, then $||V_1 - V_2||_\infty = 3$.

(10) The degree of precision in the Trapezoidal rule is 1.

Good Luck