SULTAN QABOOS UNIVERSITY

DEPARTMENT OF MATHEMATICS AND STATISTICS

Math 4141 Fall 2011

Final Exam Time: 2 hours & 30 minutes

Name:. Section: . . . . . Number. . . . .

Important Instructions

- Make sure you write your name, number and section number on the exam paper and on the solution booklet.
- Solve all questions, and make sure you show your complete, mathematically correct and neatly written solution.
- You are NOT allowed to share calculators or any other material during the test under any circumstances.
- Cellular phones are NOT allowed to be used in class.

Total Points *(80 points)*

Q1: *(3+3 points)*

(i) What does it mean when we say "a function $f(x)$ has a zero of multiplicity m at $x = p$"?
(ii) Show that $f(x) = \cos(x) - 1$ has a zero of multiplicity 2 at $x = 0$.

Q2: *(2+3 points)*

(i) What does it mean when we say "a function $f(x)$ has a fixed point at $x = p$"?
(ii) Give an example of a function that has two fixed points. Justify your answer.

Q3: *(4+4+3 points)*

(i) Show that $g(x) = \frac{1}{2}(10 - x^3)^{\frac{1}{2}}$ maps the interval $[1, 1.5]$ into itself.
(ii) Prove that g has a unique fixed point in the interval $[1, 1.5]$.
(iii) Start with $p_0 = 1$ and find p_1, p_2 and p_3 in the iteration of $p_{n+1} = g(p_n)$.

Q4: *(3+4 points)*

(i) Show that $f(x) = \frac{1}{2}(10 - x^3)^{\frac{1}{2}} - x$ has a zero in the interval $[1, 1.5]$.
(ii) Start with $p_0 = 1$ and use Newton's method to find p_1, p_2 and p_3.

Q5: *(4+5 points)*

(i) Let $a \neq b$. Interpolate the points $(a, f(a)), (b, f(b))$ with a linear polynomial $P_1(x)$.
(ii) Take $P_1(x)$ to be the linear polynomial obtained in part (i), and show that $\int_a^b P_1(x)\,dx$ gives the approximation given in the Trapezoidal rule.

Q6: *(4+5 points)*

(i) Use Simpson's rule to approximate $\int_0^1 \tan(x^2)\,dx$
(ii) Use the composite Trapezoidal rule with $n = 5$ to approximate $\int_0^1 \tan(x^2)\,dx$.

Q7: Find the constants x_0, x_1 and c_1 so that the quadrature formula *(6 points)*

$$\int_0^1 f(x)\,dx = \frac{1}{2}f(x_0) + c_1 f(x_1)$$

has the highest possible degree of precision.

Q8: Given the initial value problem *(2+5+5 points)*

$$y' = \frac{2}{t}y + t^2 e^t, \quad 1 \le t \le 2,\ y(1) = 0.$$

Answer each of the following:

(i) Show that $f(t, y) = \frac{2}{t}y + t^2 e^t$ satisfies the Lipschitz condition.
(ii) Use Taylor's method of order two with $h = 0.25$ to approximate the solution.
(iii) Use a Runge-Kutta Method (Midpoint Method) with $h = 0.25$ to approximate the solution of the initial value problem.

Q9: Given the matrix *(3+2 points)*

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 5 & 1 \\ -1 & -1 & 9 \end{bmatrix}.$$

(i) Use the Gerschgorin theorem (Gerschgorin Circle) to determine the location of the eigenvalues.
(ii) Depending on (i), how many of the eigenvalues are going to be real numbers? Justify your answer.

Q10: `(True or False questions)` Write `True` beside the true statement and `False` beside the false one in each of the following: *(1 point each)*

(1) The equation $x\cos(x) - 2x^2 + 3x - 1 = 0$ has at least one solution in the interval $[0, 1]$.

(2) The convergence in the fixed point iteration $p_{n+1} = g(p_n)$ is always linear convergence.

(3) The convergence in Newton's method is linear convergence.

(4) Aitken's Δ^2 method can be used to accelerate the convergence of a linearly convergent sequence.

(5) A polynomial of degree n has n real roots.

(6) If $x_0, x_1, \ldots, x_n$ are $n + 1$ distinct numbers and f is a function whose values are given at those numbers, then there exists a unique polynomial $P(x)$ of degree at most n such that $f(x_k) = P(x_k)$ for all $k = 0, 1, ..., n$.

(7) Simpson's rule is accurate for quadratic polynomials.

(8) The triangle with vertices $(-2, 0), (0, 3), (2, 0)$ forms a convex set in $\mathbb{R}^2$.

(9) If $V_1 = (1, 2, 3)$ and $V_2 = (3, 2, 1)$, then $||V_1 - V_2||_\infty = 3$.

(10) The degree of precision in the Trapezoidal rule is 1.

Good Luck