Sultan Qaboos University

Department of Mathematics and Statistics

Math3207 | Fall 2013

Test 2 | Time: 70 minutes

Name:. . . . . . Key. . . . . Section: . . . . ID Number:. . . . .

Group 1: Knowledge Questions

Q1: (2 points each)

(i) Write down Pascals identity?

Solution: For any positive integers n and k, the identity is

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

(ii) In how many different ways can n men sit around a circular table?

Solution: The number of arrangements is $(n-1)!$.

(iii) In how many different ways can n objects be placed around a circular table if the orientation is not important?

Solution: The number of arrangements is $\frac{1}{2}(n-1)!$.

Group 2: Comprehension & Application Questions

Q2: (2+3 points)

(i) How many arrangements can be made of the letters in `arrangements`?

`Solution:` The number of arrangements is $\frac{12!}{2!2!2!2!} = \frac{12!}{16}$.

(ii) How many arrangements can be made of the letters in `arrangements` if we want exactly 3 letters between the two letters `r`?

`Solution:` First, we arrange the letters other than the two letter r. It can be done in $\frac{10!}{2!2!2!}$. Now, we glue the two letters r with three spaces between them, then count the options. We have 8 options given by

r---r-------   -r---r------   --r---r-----   ---r---r----

----r---r---   -----r---r--   ------r---r-   -------r---r

The total number of arrangements is

$$\frac{10!}{2!2!2!}.8 = 10!$$

Q3: Find the term that is independent of x in the the expansion of (3 points)

$$\left(x - \frac{1}{x^2}\right)^{15}.$$

`Solution:` Because

$$\left(x - \frac{1}{x^2}\right)^{15} = \sum_{j=0}^{15} \binom{15}{j} x^j (-x^{-2})^{15-j} = \sum_{j=0}^{15} \binom{15}{j} (-1)^{15-j} x^{3j-30},$$

then $j = 10$ gives us the term that is independent of x. Thus, the coefficient at $j = 10$ is

$$\binom{15}{10}(-1)^5 = -\binom{15}{10}.$$

Group 3: Analysis & Application Questions

Q4: What is the coefficient of x^{10} in the expansion of (4 points)

$$(1-x^2)(1+2x)^{20}.$$

`Solution:` We write

$$(1-x^2)(1+2x)^{20} = (1-x^2)\sum_{j=0}^{20}\binom{20}{j}(2x)^j = \sum_{j=0}^{20}\binom{20}{j}2^j x^j - \sum_{j=0}^{20}\binom{20}{j}2^j x^{j+2}.$$

We obtain x^{10} in the first sum when $j=10$, and we obtain x^{10} in the second sum when $j=2$. Thus, the coefficient of x^{10} is given by

$$2^{10}\binom{20}{10} - 2^8\binom{20}{8}.$$

Q5: Prove that (4 points)

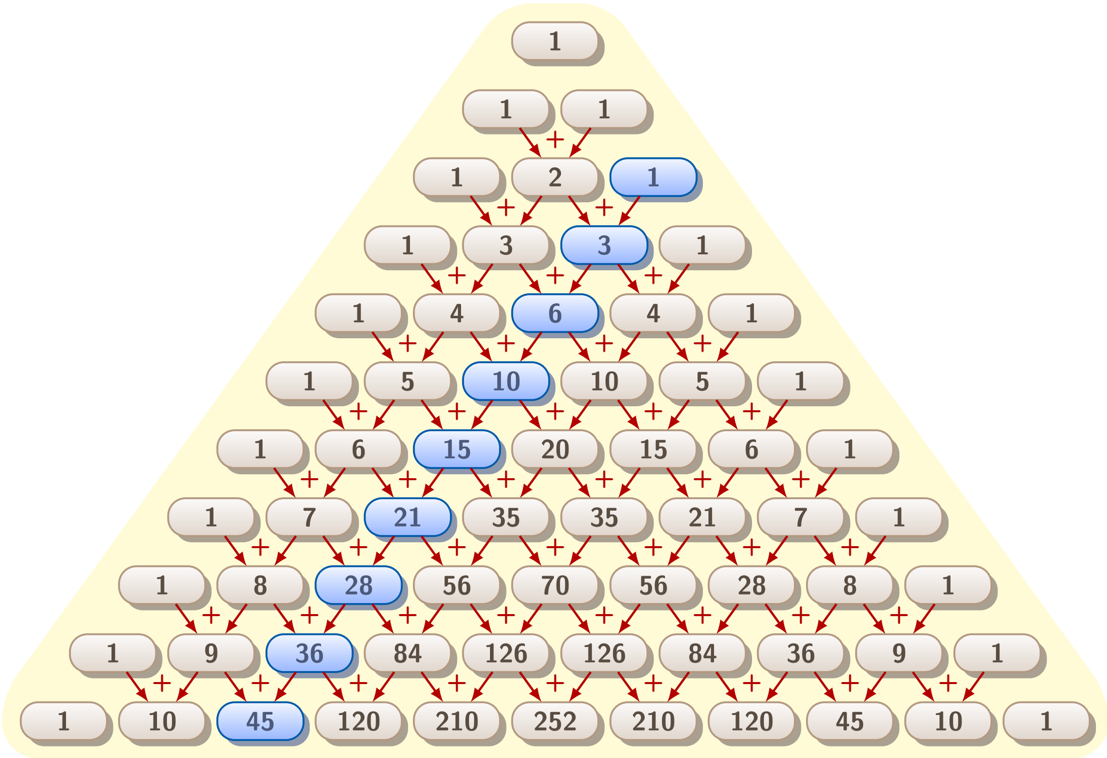
$$\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \binom{n+2}{k}.$$

`Solution:` Using Pascal's identity, we have

$$\begin{aligned}\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} &= \left(\binom{n}{k} + \binom{n}{k-1}\right) + \left(\binom{n}{k-1} + \binom{n}{k-2}\right)\\ &= \binom{n+1}{k} + \binom{n+1}{k-1}\\ &= \binom{n+2}{k}\\ &= R.H.S.\end{aligned}$$

Group 4: Evaluation Questions

Q6: Consider the following highlighted numbers in Pascal's Triangle: (3+5 points)



(i) Consider $P(x) = (1+x)^n$. Show that the highlighted numbers can be represented by

$$\binom{n}{2}, \quad n = 2, 3, \ldots$$

`Solution:` The first row from the top gives the coefficients in the expansion of $(1+x)^0$, the second row gives the coefficients in the expansion of $(1+x)^1$. In general, the nth row gives the coefficients in the expansion of $(1+x)^{n-1}$. In the $n+1$-row $(n = 2, 3, ...)$, the highlighted numbers give the coefficient of x^2 in the expansion of $(1+x)^n$. From the binomial expansion, the coefficient of x^2 is

$$\binom{n}{2}, \quad n = 2, 3, \ldots$$

(ii) Find a general formula for the sum of the highlighted numbers.

Hint: Use mathematical induction to prove that

$$\sum_{j=2}^{n} \binom{j}{2} = \frac{1}{6}n(n^2-1), \quad n = 2, 3, \ldots$$

`Proof:` We use mathematical induction to prove the formula in part (ii).
`Step One:` At $n = 2$, we have

$$\sum_{j=2}^{2} \binom{j}{2} = \binom{2}{2} = 1 \qquad \text{also} \qquad \frac{1}{6}2(2^2 - 1) = 1.$$

So, the formula is true for $n = 2$.
`Step Two:` Assume the formula is true for $n = 2, 3, ..., k$, we need to show it is true for $n = k+1$.
`Step Three:` For $n = k + 1$, we have

$$\sum_{j=2}^{k+1} \binom{j}{2} = \sum_{j=2}^{k} \binom{j}{2} + \binom{k+1}{2} = \frac{1}{6}k(k^2 - 1) + \binom{k+1}{2}.$$

Now, simplify to obtain

$$\frac{1}{6}k(k^2 - 1) + \frac{(k+1)!}{2(k-1)!} = \frac{1}{6}k(k^2 - 1) + \frac{(k+1)k}{2} = \frac{1}{6}k(k-1)(k+1) + \frac{(k+1)k}{2}.$$

Factor the right hand side to obtain

$$\frac{1}{6}k(k+1)(k-1+3) = \frac{1}{6}k(k+1)(k+2) = \frac{1}{6}(k+1)((k+1)^2 - 1),$$

which is the needed formula. Hence the formula is valid for all $n = 2, 3, \ldots$

End of Solution