Sultan Qaboos University
Department of Mathematics and Statistics

Math3207 Fall 2013

Test 1 Time: 90 minutes

Name:. . . . . . . .Key . . . . Section: . . . . ID Number:. . . . .

Group 1: Knowledge Questions

Q1: (3 points)

What does it mean when we say a set of vectors forms basis for $\mathbf{R}^2$?

Solution: A set of vectors $\mathcal{B}$ forms basis for $\mathbf{R}^2$ if the vectors of $\mathcal{B}$ are linearly independent and they span $\mathbf{R}^2$.

Q2: (2 points)

What does it mean when we say a transformation $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ is linear?

Solution: A transformation $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ is linear if it satisfies

$$T(\alpha V + \beta U) = \alpha T(V) + \beta T(U)$$

for all scalars $\alpha, \beta \in \mathbb{R}$ and all vectors $U, V \in \mathbf{R}^2$.

Group 2: Comprehension Questions

Q3: (3 points)

Which of the following vectors are along the line $5x - 3y = 0$?

$$V_1 = [3, 5], \quad V_2 = [5, 3], \quad V_3 := [-3, -5], \quad V_4 = [15, 25].$$

Solution: A vector of the form $[a, b]$ lies along a line passing through the origin if the point (a, b) satisfies the equation of the line. Because $(3, 5), (-3, -5)$ and $(15, 25)$ satisfy the equation $5x - 3y = 0$, then V_1, V_3 and V_4 are along the line. On the other hand, because the point $(5, 3)$ does not satisfy the equation $5x - 3y = 0$, then the vector V_2 does not lie along the line.

Q4: (3 points)

Which of the following vectors are perpendicular to the line $5x - 3y = 0$?

$$V_1 = [5, -3], \quad V_2 = [3, 5], \quad V_3 := [-5, 3], \quad V_4 = [50, -30].$$

Solution: Because the vector $V = [3, 5]$ lies along the line $5x - 3y = 0$, then all vector perpendicular to the line must be perpendicular to V, which means the dot product with V must be zero. Now, we have

$$V \cdot V_1 = 0, \quad V \cdot V_2 \neq 0, \quad V \cdot V_3 = 0, \quad V \cdot V_4 = 0.$$

Therefore, V_1, V_3 and V_4 are perpendicular to the line $5x - 3y = 0$.

Group 3: Application Questions

Q5: (2 points)

Find the projection of $V_1 = [1, 4]$ in the direction of $V_2 = [1, 1]$.

Solution:

$$Proj_{V_2}(V_1) = \frac{V_1 \cdot V_2}{\|V_2\|^2} V_2 = \frac{1+4}{2}[1,1] = \left[\frac{5}{2}, \frac{5}{2}\right].$$

Q6: (3 points)

Find the reflection of the point $(-1, 2)$ about the line $y = -4x$.

Solution: We take the vector $V = [-1, 2]$ and another vector along the line, say $U = [1, -4]$. Now, we find the reflection of V as

$$\begin{aligned} T(V) =& 2\frac{V \cdot U}{\|U\|^2} U - V \\ =& 2\frac{-1-8}{17}[1,-4] - [-1,2] \\ =& \frac{-18}{17}[1,-4] - [-1,2] \\ =& \left[\frac{-1}{17}, \frac{38}{17}\right]. \end{aligned}$$

Thus, the reflection of the point $(-1, 2)$ about the line $y = -4x$ is the point $\left(\frac{-1}{17}, \frac{38}{17}\right)$.

Q7: (2 points)

Find the distance between the point $(1, 2)$ and the line $3x + y = 0$.

Solution: The distance is given by

$$d = \frac{|ax_0 + by_0|}{\sqrt{a^2 + b^2}},$$

where (x_0, y_0) is the point $(1, 2)$ and $[a, b]$ is a vector perpendicular to the line. Since the vector $[3, 1]$ is perpendicular to the line, then

$$d = \frac{|3+2|}{\sqrt{9+1}} = \frac{5}{\sqrt{10}}.$$

Group 4: Analysis Questions

Q8: (3 points)

Given a point $X_0 = (x_0, y_0)$ and a vector $V = [a, b]$. Is it true that

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \end{cases}$$

gives a parametric representation of the equation of the line that passes through X_0 and parallel to V? Justify your answer.

Solution: Yes, it is true and we prove it as follows: Let (x, y) be arbitrary point on the line that passes through X_0 and parallel to V. Now, the vector that connects (x_0, y_0) to (x, y) must be $[x - x_0, y - y_0]$. Since this vector is parallel to V, then we must have

$$[x - x_0, y - y_0] = tV = [ta, tb]$$

for some scalar t. Thus, we obtain

$$x = x_0 + ta \qquad \text{and} \qquad y = y_0 + tb.$$

Since (x, y) is arbitrary point, t is arbitrary parameter and that given a representation of the equation of the line.

Group 5: Synthesis Questions

Q9: (4 points)

The points $A = (1, 2), B = (1, 8), C = (7, 2)$ and $D = (7, 8)$ are the vertices of a polygon.

(i) Use vectors to show that the polygon is in fact a square.

Solution: Connect the consecutive vertices of the polygon by vectors as follows

$$V_1 = \overrightarrow{AB} = [0, 6], \quad V_2 = \overrightarrow{BD} = [6, 0], \quad V_3 = \overrightarrow{DC} = [0, -6], \quad V_4 = \overrightarrow{CA} = [-6, 0]$$

All these vectors have length equal 6 units. Furthermore, because

$$V_1 \cdot V_2 = 0, \quad V_2 \cdot V_3 = 0, \quad V_3 \cdot V_4 = 0, \quad V_4 \cdot V_0 = 0,$$

then the angles of the polygon are right angles. Therefore, the polygon is a square.

(ii) Find a transformation T that shifts this square and makes it centered at the origin.

Solution: From part (i), the polygon is a square. Therefore, we can take the midpoints of the sides to find that $(4, 5)$ is the center of the square. Now, the transformation T that shifts this square and makes it centered at the origin must be given by

$$T([x, y]) = [x - 4, y - 5].$$

Group 6: Evaluation Questions

Q10: (5 points)

Show that the shear transformation

$$T([x, y]) = [x + 5y, y + 5x]$$

is a linear transformation, then find the matrix that can represent it.

Solution: Let $[x, y], [u, v]$ be any two vectors in $\mathbf{R}^2$ and α, β be any two scalars in $\mathbb{R}$. We have

$$\begin{aligned} T(\alpha[x, y] + \beta[u, v]) =& T([\alpha x + \beta u, \alpha y + \beta v]) \\ =& [\alpha x + \beta u + 5(\alpha y + \beta v), \alpha y + \beta v + 5(\alpha x + \beta u)] \\ =& [\alpha x + 5\alpha y, \alpha y + 5\alpha x] + [\beta u + 5\beta v, \beta v + 5\beta u] \\ =& \alpha[x + 5y, y + 5x] + \beta[u + 5v, v + 5u] \\ =& \alpha T([x, y]) + \beta T([u, v]). \end{aligned}$$

Thus, the transformation is linear.

Because

$$T([1, 0]) = [1, 5] \quad \text{and} \quad T([0, 1]) = [5, 1],$$

then these two vectors are the columns of the matrix associated with T, i.e., the matrix is

$$\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}.$$

End of Compulsory Questions

Group 7: Bonus Question (3 points)

Prove that a linear transformation $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ maps a line into a line or a point.