Sultan Qaboos University
Department of Mathematics and Statistics

Math3207 Fall 2013

Quiz 3 Time: 30 minutes

Name:. . . . . Key. . . . . Section: . . . . ID Number:. . . . .

Find all solutions (real or complex) of the following equations: *(4+5+6 points)*

(i) $x^3 = i$ (ii) $x^4 - 6x^3 + 9x^2 - 1 = 0$ (iii) $x^3 + x^2 - 9x - 1010 = 0.$

(i) $x^3 = i$

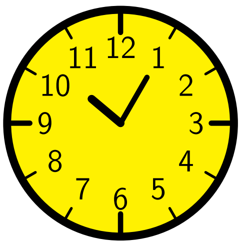
Solution:

We use De Moivre's Formula

$$x_k = |z_0|^{\frac{1}{n}} \left(\cos\left(\frac{\theta_0 + 2k\pi}{n}\right) + i\sin\left(\frac{\theta_0 + 2k\pi}{n}\right)\right), \quad k = 0, 1, 2, \ldots, n-1$$

to find the three solutions. Thus,

$$\begin{aligned} x_k =& |i|^{\frac{1}{3}} \left(\cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)\right), \quad k = 0, 1, 2. \\ x_0 =& \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i. \\ x_1 =& \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2} + \frac{1}{2}i. \\ x_2 =& \cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right) = -i. \end{aligned}$$



(ii) $x^4 - 6x^3 + 9x^2 - 1 = 0.$

Solution:

We write the equation as $x^4 - 6x^3 = -9x^2 + 1$, then make the L.H.S a perfect square to obtain

$$(x^2 - 3x)^2 = -9x^2 + 1 + 9x^2 = 1.$$

Thus, we obtain a difference of two squares that can be factored as

$$(x^2 - 3x - 1)(x^2 - 3x + 1) = 0.$$

$x^2 - 3x - 1 = 0$ gives us

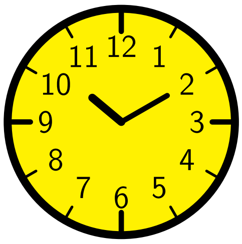
$$x = \frac{1}{2}\left(3+\sqrt{13}\right) \qquad \text{or} \qquad x = \frac{1}{2}\left(3-\sqrt{13}\right).$$

Also, $x^2 - 3x + 1 = 0$ gives us

$$x = \frac{1}{2}\left(3+\sqrt{5}\right) \qquad \text{or} \qquad x = \frac{1}{2}\left(3-\sqrt{5}\right).$$

Hence, the four solutions are real and given by

$$\frac{1}{2}\left(3+\sqrt{13}\right), \quad \frac{1}{2}\left(3-\sqrt{13}\right), \quad \frac{1}{2}\left(3+\sqrt{5}\right), \quad \frac{1}{2}\left(3-\sqrt{5}\right).$$



(iii) $x^3 + x^2 - 9x - 1010 = 0$.

`Solution:`
We use Cardano's method (also known as Tartaglia's method) as follows:
Let $x = t - \frac{1}{3}$, then we obtain

$$(t-\frac{1}{3})^3 + (t-\frac{1}{3})^2 - 9t - 1007 = 0,$$

which simplifies to

$$t^3 - \frac{28}{3}t - \frac{27187}{27} = 0.$$

Now, we have a cubic equation of the form $t^3 + pt + q = 0$. Let $t = u + v$, we obtain

$$u^3 + v^3 + (u+v)(3uv - \frac{28}{3}) - \frac{27187}{27} = 0.$$

Next, we have the freedom to force

$$uv = \frac{28}{9},$$

which means that we have to solve the system

$$\begin{cases} u^3 + v^3 = \frac{27187}{27} \\ u^3v^3 = \frac{28^3}{9^3}. \end{cases}$$

Next, think about $u^3 = \alpha$ and $v^3 = \beta$, then $\alpha + \beta$ and $\alpha\beta$ are the roots of the quadratic equation

$$z^2 - (\alpha+\beta)z + \alpha\beta = 0.$$

Thus, we have to solve

$$z^2 - \frac{27187}{27}z + \frac{28^3}{9^3} = 0,$$

which gives us

$$\alpha = u^3 = \frac{27187}{54} + \frac{311}{18}\sqrt{849} \quad \text{and} \quad \beta = v^3 = \frac{27187}{54} - \frac{311}{18}\sqrt{849}.$$

Next, we take

$$\alpha^{\frac{1}{3}} = \left(\frac{27187}{54} + \frac{311}{18}\sqrt{849}\right)^{\frac{1}{3}} = c_1 + c_2 \quad \text{and} \quad \beta^{\frac{1}{3}} = \left(\frac{27187}{54} - \frac{311}{18}\sqrt{849}\right)^{\frac{1}{3}} = c_1 - c_2,$$

where we simplify our writing by taking $c_1 = \frac{31}{6}$ and $c_2 = \frac{1}{6}\sqrt{849}$. Therefore,

$$u = \alpha^{\frac{1}{3}}, \alpha^{\frac{1}{3}}\omega, \alpha^{\frac{1}{3}}\omega^2 \quad \text{and} \quad v = \beta^{\frac{1}{3}}, \beta^{\frac{1}{3}}\omega, \beta^{\frac{1}{3}}\omega^2.$$

The value of $t = u + v$ is given by

$$\begin{aligned} t_1 &= \left(\alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}}\right) = \frac{31}{3} \\ t_2 &= (c_1 + c_2)\omega + (c_1 - c_2)\omega^2 = c_1(\omega + \omega^2) - c_2(\omega^2 - \omega) = -c_1 + c_2\sqrt{3}i \\ t_3 &= (c_1 + c_2)\omega^2 + (c_1 - c_2)\omega = c_1(\omega^2 + \omega) + c_2(\omega^2 - \omega) = -c_1 - c_2\sqrt{3}i \end{aligned}$$

Finally, the solutions of the original equation are given by $x = t - \frac{1}{3}$, which are

$$\begin{aligned} x_1 &= \frac{31}{3} - \frac{1}{3} = 10 \\ x_2 &= -c_1 + c_2\sqrt{3}i - \frac{1}{3} = \frac{-31}{6} - \frac{1}{3} + \frac{\sqrt{3}\sqrt{849}}{6} = -\frac{11}{2} + \frac{\sqrt{283}}{2}i \\ x_3 &= -c_1 - c_2\sqrt{3}i - \frac{1}{3} = \frac{-31}{6} - \frac{1}{3} - \frac{\sqrt{3}\sqrt{849}}{6} = -\frac{11}{2} - \frac{\sqrt{283}}{2}i. \end{aligned}$$