Sultan Qaboos University
Department of Mathematics and Statistics

Math3207 Fall 2013

Final Exam Wednesday, January 15, 2014 Time: 150 minutes

Name:. . . . .Key. . . . . Section: . . . . ID Number:. . . . .

Q1: Give a concise meaning (definition) of each of the following:

(i) A single variable polynomial of degree n;

`Solution:` A function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0,$$

where $a_n \neq 0$ and the coefficients are constants is called a single variable polynomial of degree n.

(ii) A quartic equation in one unknown;

`Solution:` An equation of the form

$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$

where $a \neq 0$ and a, b, c, d, e are constants is called a quartic equation in one unknown.

(iii) Linearly dependent set of vectors;

`Solution:` A set of vectors is called linearly dependent set if some elements of the set can be obtained by a linear combination of other elements in the set.

(iv) A parallelogram;

`Solution:` A parallelogram is a quadrilateral in which each two opposite sides are parallel and equal in length.

(v) A unit vector.

`Solution:` A unit vector is a vector of length one.

Q2: Give a complete statement of each of the following theorems.

(i) Binomial Theorem;

`Solution:` For a non-negative integer n, the Binomial Theorem states that

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}.$$

(ii) Fundamental Theorem of Algebra.

`Solution:` Every non-constant single-variable polynomial with complex or real coefficients has at least one complex root.

Q3: In how many different ways can 10 people be seated around a circular table if two of them (a man and his wife) want to sit together?

`Solution:` Think about the man and his wife as one person, then we have 9 people that must be seated around a circular table. This can be done in 8!. Now, because we have two options for the man and his wife (one of them to the left or to the right of the other), then the total number of ways is

$$8! \times 2.$$

Q4: Give an example of a line and a vector so that the vector is perpendicular to the line. The equation of the line must be of the form $y = mx + b$. Also, make sure that you justify your answer.

`Solution:` If we take the line $y = x$ and the vector $V = [-1, 1]$, then a vector parallel to $y = x$ is $U = [1, 1]$. Because

$$U \cdot V = -1 + 1 = 0,$$

then the two vectors are perpendicular. Hence, V is perpendicular to the line $y = x$.

Q5: Give an example of a cubic equation in the form $x^3 + bx^2 + cx + d = 0$ such that the equation has a zero of multiplicity three at $x = 2$..

`Solution:` Since $x = 2$ is zero of multiplicity three, then $(x-2)^3$ is a factor. However, when we expand, we obtain the required equation, which is given by

$$(x-2)^3 = x^3 - 6x^2 + 12x - 8 = 0.$$

Q6: Find the projection of the vector $V = [-2, 3]$ in the direction of the vector $U = [1, -2]$.

`Solution:`

$$Proj_U(V) = \frac{U \cdot V}{\|U\|^2} U = \frac{-2-6}{5}[1, -2] = \left[\frac{-8}{5}, \frac{16}{5}\right].$$

Q7: Find the middle term in the expansion of $(x + \frac{2}{x})^{50}$.

`Solution:` The expansion is given by

$$\left(x + \frac{2}{x}\right)^{50} = \sum_{k=0}^{50} \binom{50}{k} x^k \left(\frac{2}{x}\right)^{50-k}.$$

The middle term is obtained at $k = 25$. Thus, the middle term is

$$\binom{50}{25} x^{25} \left(\frac{2}{x}\right)^{25} = \binom{50}{25} 2^{25}.$$

Q8: Find all solutions of the equation $x^6 = 1 + i$.

Solution: We use De-Moivre's Formula

$$x_k = |z_0|^{\frac{1}{n}} \left(\cos\left(\frac{\theta_0 + 2k\pi}{n}\right) + i \sin\left(\frac{\theta_0 + 2k\pi}{n}\right) \right), \quad k = 0, 1, 2, \dots, n-1$$

to find the six solutions. Because $\theta_0 = \frac{\pi}{4}$ and $|z_0| = \sqrt{2}$, then,

$$\begin{aligned}
x_k =& |2|^{\frac{1}{12}} \left(\cos\left(\frac{\frac{\pi}{4} + 2k\pi}{6}\right) + i \sin\left(\frac{\frac{\pi}{4} + 2k\pi}{6}\right) \right), \quad k = 0, 1, 2, 3, 4, 5. \\
x_0 =& 2^{\frac{1}{12}} \left(\cos\left(\frac{\pi}{24}\right) + i \sin\left(\frac{\pi}{24}\right) \right) \\
x_1 =& 2^{\frac{1}{12}} \left(\cos\left(\frac{9\pi}{24}\right) + i \sin\left(\frac{9\pi}{24}\right) \right) = 2^{\frac{1}{12}} \left(\cos\left(\frac{3\pi}{8}\right) + i \sin\left(\frac{3\pi}{8}\right) \right) \\
x_2 =& 2^{\frac{1}{12}} \left(\cos\left(\frac{17\pi}{24}\right) + i \sin\left(\frac{17\pi}{24}\right) \right) \\
x_3 =& 2^{\frac{1}{12}} \left(\cos\left(\frac{25\pi}{24}\right) + i \sin\left(\frac{25\pi}{24}\right) \right) \\
x_4 =& 2^{\frac{1}{12}} \left(\cos\left(\frac{33\pi}{24}\right) + i \sin\left(\frac{33\pi}{24}\right) \right) = 2^{\frac{1}{12}} \left(\cos\left(\frac{11\pi}{8}\right) + i \sin\left(\frac{11\pi}{8}\right) \right) \\
x_5 =& 2^{\frac{1}{12}} \left(\cos\left(\frac{41\pi}{24}\right) + i \sin\left(\frac{41\pi}{24}\right) \right).
\end{aligned}$$

Q9: Find the base 3 representation of the denary number 3207.

Solution:

Step 1: Divide 3207 by 3 to obtain 1069 and remainder 0.

Step 2: Divide 1069 by 3 to obtain 356 and remainder 1.

Step 3: Divide 356 by 3 to obtain 118 and remainder 2.

Step 4: Divide 118 by 3 to obtain 39 and remainder 1.

Step 5: Divide 39 by 3 to obtain 13 and remainder 0.

Step 6: Divide 13 by 3 to obtain 4 and remainder 1.

Step 7: Divide 4 by 3 to obtain 1 and remainder 1.

Step 8: Divide 1 by 3 to obtain 0 and remainder 1.

Hence, the answer in base 3 is 11101210_3.

Q10: Without changing into denary form, find the result of $351_6 \times 345_6$.

Solution:

$$\begin{aligned}
351_6 \times 345_6 =& 1_6 \times 345_6 + 50_6 \times 345_6 + 300_6 \times 345_6 \\
=& 345_6 + 5_6 \times 3450_6 + 3_6 \times 34500_6 \\
=& 345_6 + 31010_6 + 152300_6 \\
=& 224055_6.
\end{aligned}$$

Q11: Show that $x = 2$ is a solution of

$$x^6 - 9x^5 + 30x^4 - 40x^3 + 48x - 32 = 0$$

with multiplicity 5.

`Solution:` Let $f(x) = x^6 - 9x^5 + 30x^4 - 40x^3 + 48x - 32$. We have $f(2) = 0$. Also,

$$\begin{aligned}
f'(x) &= 6x^5 - 45x^4 + 120x^3 - 120x^2 + 48 && \Rightarrow && f'(2) = 0 \\
f''(x) &= 30x^4 - 180x^3 + 360x^2 - 240x && \Rightarrow && f''(2) = 0 \\
f'''(x) &= 120x^3 - 540x^2 + 720x - 240 && \Rightarrow && f'''(2) = 0 \\
f''''(x) &= 360x^2 - 1080x + 720 && \Rightarrow && f''''(2) = 0 \\
f'''''(x) &= 720x - 1080 && \Rightarrow && f'''''(2) = 360 \neq 0
\end{aligned}$$

Because $f^{(k)}(2) = 0$ for $k = 0, 1, 2, 3, 4$ and $f^{(5)}(2) \neq 0$, then $x = 2$ is a zero of multiplicity 5.

Q12: Let ω^* be a non-real solution of the equation $x^3 + a^3 = 0$, where a is a real number. Show that $\omega^{*2} + a^2 = a\omega^*$.

`Solution:` Because ω^* is a solution of $x^3 + a^3 = 0$, then

$$\omega^{*3} + a^3 = (\omega^* + a)(\omega^{*2} - a\omega^* + a^2) = 0.$$

Because ω^* is not real, then $\omega^* + a \neq 0$, and consequently

$$\omega^{*2} - a\omega^* + a^2 = 0.$$

Thus, we obtain $\omega^{*2} + a^2 = a\omega^*$ as required.

Q13: Is it true that a linear transformation must take the zero vector into itself? Justify your answer.

`Solution:` Yes, it is true. Let T be a linear transformation, then

$$T(\overrightarrow{0}) = T(\alpha\overrightarrow{0}) = \alpha T(\overrightarrow{0}).$$

Now, take the scalar α to be zero, then we obtain $T(\overrightarrow{0}) = \overrightarrow{0}$.

Q14: Show that a shear transformation $T([x, y]) = [x + ky, y + rx]$ must map a line segment into a line segment.

`Solution:` Using a parametric form of the line equation, i.e., $x = x_0 + at, y = y_0 + bt$, we use the linearity of the transformation to obtain

$$T([x_0 + at, y_0 + bt]) = T([x_0, y_0] + t[a, b]) = T([x_0, y_0]) + tT([a, b]).$$

Now, call $T([x_0, y_0]) = [x_0 + ky_0, y_0 + rx_0] = [X_0, Y_0]$ and $T([a, b]) = [a + kb, b + ra] = [A, B]$, then

$$[X_0, Y_0] + t[A, B] = [X_0 + tA, Y_0 + tB],$$

which gives an equation of a line in parametric form.

Q15: For all positive integers n, prove that

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k}(-1)^k = 1.$$

`Solution:` From the Binomial theorem,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

If, we take $x = 2$, and $y = -1$, then

$$(2-1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}(-1)^k.$$

Hence,

$$1 = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}(-1)^k$$

as required.