Sultan Qaboos University

Department of Mathematics and Statistics

Math3110 | Fall 2014

Test I | Time: 60 minutes

Name:. . . . . Key . . . .   Section: . . . . .   Number. . . . .

Q1: (a) Find a vector of magnitude 4 that is orthogonal to $V =< 3, 0, 4 >$. *(3 points)*

Solution:

Because $< -4, 0, 3 > \cdot < 3, 0, 4 >= 0$, then $U =< -4, 0, 3 >$ is orthogonal to V. Now, $U_2 = \frac{4}{5} < -4, 0, 3 >$ is orthogonal to V and has length equals 4.

(b) Given $U =< 0, -3, 4 >$ and $V =< 2, 0, -2 >$. Find $Comp_U(V)$. *(2 points)*

Solution:

$$Comp_U(V) = \frac{U \cdot V}{\|U\|} = \frac{-8}{5}.$$

Q2: *(3 + 4 points)*

(a) Are the vectors $V =< 1, -3, 1 >$, $U =< 2, -1, 0 >$ and $W =< 0, -5, 2 >$ coplanar (belong to the same plane)? Justify your answer.

Solution: If $(V \times U) \cdot W = 0$, then $V \times U$ is perpendicular to V, U and W, and consequently the three vectors are coplanar. Now, we have

$$(V \times U) \cdot W = \begin{vmatrix} 0 & -5 & 2 \\ 1 & -3 & 1 \\ 2 & -1 & 0 \end{vmatrix} = 0 + 5(0 - 2) + 2(-1 + 6) = 0.$$

Hence, the three vectors must be coplanar.

(b) Find an equation of the plane containing the point $(3, -2, 1)$ and parallel to the plane $x + 3y - 4z = 2$.

Solution:

Because the vector $U =< 1, 3, -4 >$ is normal to both planes, then we use U and the point to find the equation. Indeed, we obtain

$$(x - 3) + 3(y + 2) - 4(z - 1) = 0.$$

Q3: For any three vectors U, V_1 and V_2 in space, show that *(4 points)*

$$Proj_U(V_1 + V_2) = Proj_U(V_1) + Proj_U(V_2).$$

Proof:

$$\begin{aligned} Proj_U(V_1 + V_2) &= \frac{(V_1 + V_2) \cdot U}{\|U\|} \frac{U}{\|U\|} \\ &= \frac{V_1 \cdot U + V_2 \cdot U}{\|U\|} \frac{U}{\|U\|} \\ &= \frac{V_1 \cdot U}{\|U\|} \frac{U}{\|U\|} + \frac{V_2 \cdot U}{\|U\|} \frac{U}{\|U\|} \\ &= Proj_U(V_1) + Proj_U(V_2). \end{aligned}$$

Q4: Identify the graph (NO need to sketch) of each of the following in space *(6 points)*
(i) $x^2 + y^2 = 4$ (ii) $x - y = 1$ (iii) $3x^2 + 4y^2 + 5z^2 = 10.$

`Solution:`

(i) We obtain a cylinder
(ii) We obtain a plane
(iii) We obtain an ellipsoid.

Q5: (a) Evaluate *(3 + 4 points)*

$$\lim_{t\to\infty} < te^{-t}, \frac{t^3+t}{2t^3-1}, t\sin\left(\frac{1}{t}\right) > .$$

`Solution:`

Because

$$\lim_{t\to\infty} te^{-t} = \lim_{t\to\infty} \frac{t}{e^t} = 0,$$

$$\lim_{t\to\infty} \frac{t^3+t}{2t^3-1} = \frac{1}{2}$$

and

$$\begin{aligned} \lim_{t\to\infty} t\sin\left(\frac{1}{t}\right) &= \lim_{t\to\infty} \frac{\sin\left(\frac{1}{t}\right)}{\frac{1}{t}} \\ &= \lim_{T\to 0^+} \frac{\sin(T)}{T} \\ &=1. \end{aligned}$$

Hence,

$$\lim_{t\to\infty} < te^{-t}, \frac{t^3+t}{2t^3-1}, t\sin\left(\frac{1}{t}\right) > = < 0, \frac{1}{2}, 1 > .$$

(b) Find the antiderivative of $r'(t) = < \cos(t), e^{-t}, \sqrt{t} >$ satisfying the initial condition $r(0) = < 1, 2, 3 >$.

`Solution:` Because

$$r(t) - r(0) = \int_0^t < \cos(s), e^{-s}, \sqrt{s} > ds = < \sin(t), -e^{-t}, \frac{2}{3}t^{\frac{3}{2}} > \; - \; < 0, -1, 0 >,$$

then

$$r(t) = < \sin(t), -e^{-t}, \frac{2}{3}t^{\frac{3}{2}} > \; - \; < 0, -1, 0 > + < 1, 2, 3 > = < \sin(t) + 1, -e^{-t} + 3, \frac{2}{3}t^{\frac{3}{2}} + 3 > .$$

Q6: *(3 + 4 points)*

(a) Find a parametric representation of the elliptic paraboloid $z = x^2 + 4y^2$.

`Solution:` Consider $x = r\cos(\theta)$ and $y = \frac{1}{2}r\sin(\theta)$, then $z = r^2$. Thus, a parametric representation can be

$$\begin{cases} x = r\cos(\theta) \\ y = \frac{1}{2}r\sin(\theta) \\ z = r^2. \end{cases}$$

(b) At what points does the curve of $r(t) =< t, 0, 2t - t^2 >$ intersect the paraboloid $z = x^2 + y^2$?

`Solution:` We substitute $x = t, y = 0$ and $z = 2t - t^2$ in the equation $z = x^2 + y^2$ to obtain

$$2t - t^2 = t^2 + 0.$$

We solve the equation to obtain $t = 0$ or $t = 1$. Thus, the points of intersection are

$$(0, 0, 0) \qquad \text{and} \qquad (1, 0, 1).$$

Q7: Let $r(t)$ be a differentiable vector valued function. Is it true that *(4 points)*

$$\frac{d}{dt}\|r(t)\| = \|r'(t)\|?$$

Justify your answer.

`Answer:` Not true, and here is a counterexample:

Consider $r(t) =< t, 0, 0 >$, then

$$\frac{d}{dt}\|r(t)\| = \frac{d}{dt}\sqrt{t^2} = \frac{d}{dt}|t| = \begin{cases} 1, & t > 0 \\ \text{not differentiable}, & t = 0 \\ -1, & t < 0 \end{cases}.$$

On the other hand,

$$\|r'(t)\| = \| < 1, 0, 0 > \| = 1.$$

`Best Wishes` `Total: 40 points`