SULTAN QABOOS UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS

Math3110 Fall 2014

Test II Time: 60 minutes

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Name:. . . . . . . . . . . . . . . . . . . . Section: . . . . . Number. . . . . . . .

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**Important Instructions**

- Write your name, ID # and Section # on the front cover of your answer booklet.
- You need to show your complete, mathematically correct and neatly written solution.
- You are NOT allowed to share calculators or any other material during the test.
- Cellular phones are NOT allowed to be used for any purpose during the test.
- You should NOT ask the invigilator questions about the exam.
- You need to solve questions **Q1** through **Q6**.

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**Q1:** Show that the following limit does not exist *(4 points)*

$$\lim_{(x,y)\to(0,1)} \frac{xy - x}{x^2 + (y-1)^2}.$$

**Q2:** Let $z = \sin(x)\cos(y)$, where $x = (s-t)^2$ and $y = s^2 - t^2$. *(4 points)*
Use the chain rule to find $\dfrac{\partial z}{\partial s}$.

**Q3:** Find the absolute extrema of the function $f(x, y) = x^2 + y^2 - 4xy$ on the region bounded by the curves of $y = x + 1$, $y = -3$ and $x = 3$. *(6 points)*

**Q4:** Given that *(6 points)*

$$F(x, y, z) = 200e^{-x^2-3y^2-9z^2}.$$

(i) Find the rate of change at the point $P(2, -1, 2)$ in the direction toward the point $Q(3, -3, 3)$.

(ii) Find the direction in which $F$ increases most rapidly at $P$.

(iii) Find the maximum rate of increase of $F$ at $P$.

**Q5:** Evaluate the integral *(4 points)*

$$\int_0^1 \int_x^1 e^{y^2}\, dydx.$$

**Q6:** Find the volume of the solid bounded by $z = 1 - y^2$, $x + y = 1$ and the three coordinate planes (first octant). *(6 points)*

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End of Questions Best Wishes Total: 30 points

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# Sultan Qaboos University
## Department of Mathematics and Statistics

MATH3110 Fall 2014

Test II Solutions

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**Q1:** Show that the following limit does not exist *(4 points)*

$$\lim_{(x,y)\to(0,1)} \frac{xy - x}{x^2 + (y-1)^2}.$$

Solution:

Along the $y$-axis, $f(x,y) = f(0,y) = 0$ for $y \neq 1$. Therefore

$$\lim_{(x,y)\to(0,1)} f(x,y) = 0 \text{ along the } y\text{-axis.}$$

Along the line $y = x+1$, $f(x,y) = \dfrac{x(x+1) - x}{x^2 + (x+1-1)^2} = \dfrac{x^2}{2x^2} = \dfrac{1}{2}$ for $x \neq 0$. Therefore

$$\lim_{(x,y)\to(0,1)} f(x,y) = \frac{1}{2} \text{ along the line } y = x+1.$$

Hence the limit does not exist.

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**Q2:** Let $z = \sin(x)\cos(y)$, where $x = (s-t)^2$ and $y = s^2 - t^2$. *(4 points)*

Use the chain rule to find $\dfrac{\partial z}{\partial s}$.

Solution:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial x} = \cos x \cos y, \quad \frac{\partial z}{\partial y} = -\sin x \sin y$$

$$\frac{\partial x}{\partial s} = 2(s-t), \quad \frac{\partial y}{\partial s} = 2s$$

Hence

$$\begin{aligned}\frac{\partial z}{\partial s} &= 2(s-t)\cos x \cos y - 2s \sin x \sin y \\ &= 2(s-t)\cos(s-t)^2 \cos(s^2-t^2) - 2s\sin(s-t)^2 \sin(s^2-t^2)\end{aligned}$$

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**Q3:** Find the absolute extrema of the function $f(x,y) = x^2 + y^2 - 4xy$ on the region bounded by the curves of $y = x+1$, $y = -3$ and $x = 3$. *(6 points)*

Solution:

$$f(x,y) = x^2 + y^2 - 4xy, \quad R = \{(x,y) | \quad -3 \le y \le x+1, \quad -4 \le x \le 3\}$$

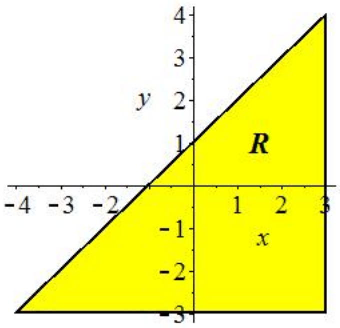


Figure 1: Region $R$ in the $xy$-plane

$$f_x = 2x - 4y, \quad f_y = -4x + 2y$$

Setting $f_x = 0, f_y = 0$ gives the critical point $(0,0) \in R$.
Now consider the function $f$ along the three boundaries $y = x+1, y = -3$ and $x = 3$.

- Along the path $y = x+1$,

$$f(x,y) = f(x,x+1) = f_1(x) = x^2 - 4x(x+1) + (x+1)^2 = -2x^2 - 2x + 1$$

$$f_1'(x) = -4x - 2, \quad f_1'(x) = 0 \Rightarrow x = -\frac{1}{2}$$

$f_1''(x) = -4 < 0$. Hence $f_1(x)$ has a maximum at $(-\frac{1}{2}, \frac{1}{2})$.

- Along the path $y = -3$,

$$f(x,y) = f(x,-3) = f_2(x) = x^2 + 12x + 9$$

$$f_2'(x) = 2x + 12, \quad f_2'(x) = 0 \Rightarrow x = -6$$

The point $(-6,-3)$ is not in the region $R$.

- Along the path $x = 3$,

$$f(x,y) = f(3,y) = f_3(y) = y^2 - 12y + 9$$

$$f_3'(y) = 2y - 12, \quad f_3'(y) = 0 \Rightarrow y = 6$$

The point $(3,6)$ is not in the region $R$.

- The points of intersection of the boundaries are $(-4,-3), \quad (3,-3), \quad (3,4)$.

Evaluating the function $f$ at all these points including the critical point $(0,0)$ :

$$f(0,0) = 0, \quad f(-\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{3}{2}, \quad f(-4,-3) = -23, \quad f(3,-3) = 54, \quad f(3,4) = -23$$

$f(x,y)$ has an absolute maximum 54 at $(3,-3)$ and an absolute minimum $-23$ at $(-4,-3)$ and $(3,4)$.

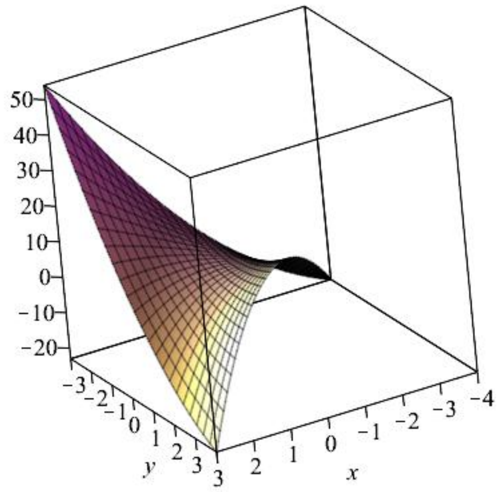


Figure 2: Graph of $f(x) = x^2 + y^2 - 4xy$ with $(x, y) \in R$

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**Q4:** Given that *(6 points)*

$$F(x,y,z) = 200e^{-x^2-3y^2-9z^2}.$$

(i) Find the rate of change at the point $P(2,-1,2)$ in the direction toward the point $Q(3,-3,3)$.

(ii) Find the direction in which $F$ increases most rapidly at $P$.

(iii) Find the maximum rate of increase of $F$ at $P$.

Solution:

(i) The rate of change of $F(x,y,z)$ at the point $P$ in the direction of the point $Q$ is given by the directional derivative of $F(x,y,z)$ in the direction of the vector $\vec{PQ}$., i.e.,

$$D_{\vec{PQ}}F = \frac{\nabla F \cdot \vec{PQ}}{||\vec{PQ}||}, \quad \text{(where } \nabla F \text{ is evaluated at } P\text{)}$$

The gradient of $F(x,y,z)$ at a point $(x,y,z)$ is given by:

$$\begin{aligned} \nabla F &= \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \\ &= -e^{-x^2-3y^2-9z^2} \langle 400x,\ 1200y,\ 3600z \rangle \end{aligned}$$

At the point $P(2,-1,2)$, the gradient $\nabla F$ is given by:

$$\nabla F = e^{-43} \langle -800,\ 1200,\ -7200 \rangle$$

The vector $\vec{PQ}$ is given by $< 1,-2,1 >$. Hence

$$\begin{aligned} D_{\vec{PQ}}F &= \frac{\nabla F \cdot \vec{PQ}}{||\vec{PQ}||} = \frac{e^{-43} \langle -800,\ 1200,\ -7200 \rangle \cdot < 1,-2,1 >}{\sqrt{6}} \\ &= \\ &= -\frac{5200}{3} e^{-43}\sqrt{6} \end{aligned}$$

(ii) The direction in which $F$ increases most rapidly at $P$ is given by the value of the gradient of $F$ at $P$, i.e. by the vector $e^{-43}\langle -800, 1200, -7200 \rangle$ or simply in the direction of the unit vector:

$$\mathbf{u} = \frac{1}{\sqrt{337}} \langle -2, 3, -18 \rangle$$

(iii ) The maximum rate of increase of $F$ at $P$ is given by:

$$\begin{aligned} ||\nabla F|| &= e^{-43}\sqrt{800^2 + 1200^2 + 7200^2} \\ &= e^{-43}\sqrt{53920000} \\ &= 400e^{-43}\sqrt{337} \end{aligned}$$

**Q5:** Evaluate the integral *(4 points)*

$$\int_0^1 \int_x^1 e^{y^2}\, dydx.$$

Solution:
The region of integration is:

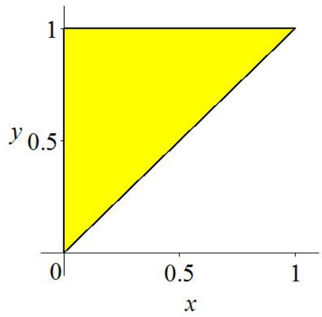


Figure 3: Region of Integration in the $xy$-plane

Changing the order of integration gives:

$$\begin{aligned}
\int_0^1 \int_x^1 e^{y^2} dydx &= \int_0^1 \int_0^y e^{y^2} dxdy \\
&= \\
&= \int_0^1 xe^{y^2}\Big|_0^y dy \\
&= \int_0^1 ye^{y^2} dy \\
&= \frac{1}{2}\, e^{y^2}\Big|_0^1 \\
&= \tfrac{1}{2}(e-1)
\end{aligned}$$

**Q6:** Find the volume of the solid bounded by $z = 1 - y^2$, $x + y = 1$ and the three coordinate planes (first octant). *(6 points)*

Solution:
The region $R$ in the $xy$-plane is given by:

$$R = \{(x, y)| \quad 0 \le y \le 1 - x, \quad 0 \le x \le 1\}$$

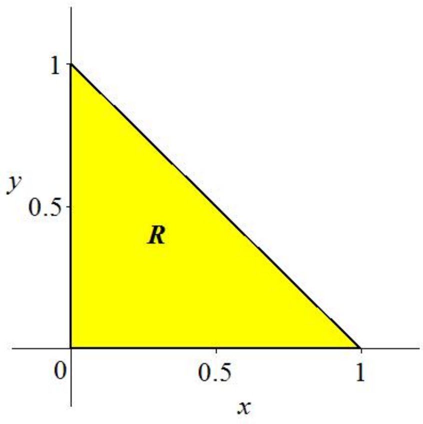


Figure 4: Region $R$ in the $xy$-plane

Let $z = f(x, y) = 1 - y^2$. Then $f$ is continuous in $R$ and $f(x, y) \ge 0$ in $R$. Therefore the volume $V$ of the solid below the surface $z = f(x, y)$ and above the region $R$ in the $xy$-plane is given by:

$$\begin{aligned} V &= \iint_R f(x,y)dA = \int_0^1 \int_0^{1-x} (1 - y^2)dydx \\ &= \int_0^1 \left(y - \frac{y^3}{3}\right)\Bigg|_0^{1-x} dx = \int_0^1 \left[1 - x - \frac{(1-x)^3}{3}\right] dx = \left[x - \frac{x^2}{2} + \frac{(1-x)^4}{12}\right]\Bigg|_0^1 \\ &= \left(1 - \frac{1}{2} + 0\right) - \left(0 - 0 + \frac{1}{12}\right) = \frac{5}{12} \end{aligned}$$

Alternatively the volume $V$ is given by the triple integral: $V = \int_0^1 \int_0^{1-x} \int_0^{1-y^2} dzdydx$

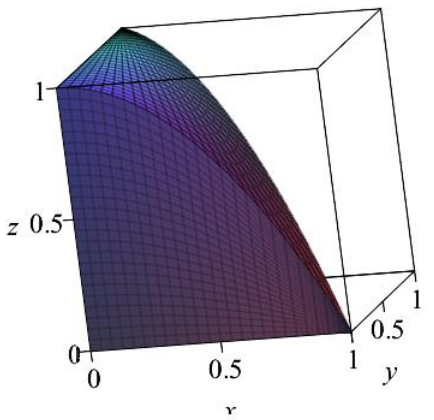


Figure 5: Graph of the solid