SULTAN QABOOS UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS

Math3110 | Fall 2014
Final Exam Solutions | Time: 150 minutes

Q1: *(4 points)*

Find all vectors of magnitude $\frac{1}{2}$ that are parallel to $V =< -6, 3, -2 >$.

Solution:
The unit vectors which are parallel to V are given by:

$$\begin{aligned} \mathbf{v} &= \pm\frac{1}{\sqrt{(-6)^2+3^2+(-2)^2}} < -6,\ 3,\ -2 > \\ &= \pm\frac{1}{7} < -6,\ 3,\ -2 > . \end{aligned}$$

Hence, the two vectors of magnitude $\frac{1}{2}$ that are parallel to V are:

$$\mathbf{u} = \frac{1}{2}\mathbf{v} = \pm\frac{1}{14} < -6,\ 3,\ -2 > .$$

Q2: *(6 points)*

Find the area of the triangle with vertices $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

Solution:
Constructing two edge vectors $\vec{PQ}$ and $\vec{PR}$ gives

$$\vec{PQ} =< 1,\ 2,\ -1 > \quad \text{and} \quad \vec{PR} =< -2,\ 2,\ 2 > .$$

The area of the triangle PQR is given by $A = \frac{1}{2}\left\|\vec{PQ}\times\vec{PR}\right\|$.

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} =< 6,\ 0,\ 6 >$$

Hence, the area $A = \frac{1}{2}\sqrt{6^2+0^2+6^2} = 3\sqrt{2}$.

Q3: *(6 points)*

Show that vector $V =< a, b >$ is orthogonal to the line $ax + by = c$.

Solution:
Let L be the the line $ax + by = c$. Then L can be expressed as a dot product:

$$V \cdot r = c,$$

where $V =< a,\ b >$ and $r =< x,\ y >$ is the position vector of a point on the line L. Now consider the line L_1 described by $ax + by = 0$ or $V \cdot r = 0$. Clearly L_1 is parallel to L since they have the same slope. From $V \cdot r = 0$, the vector V is orthogonal to L_1 and hence orthogonal to L.

Q4: *(8 points)*

Consider the two points $A(-3,3,-2)$, $B(2,-1,4)$ and the line L that passes through them.

(i) Find parametric equations that represent L.

(ii) At what point does L intersect the xy-plane?

Solution:

(i) A vector that is parallel to L is given by

$$\vec{AB} =< 2-(-3),\ -1-3,\ 4-(-2) >=< 5,\ -4,\ 6 > .$$

Using the point A and the vector $\vec{AB}$ gives a parametric representation of L:

$$\begin{aligned} r(t) &= < x(t),\ y(t),\ z(t) > \\ &= < -3,\ 3,\ -2 > + t < 5,\ -4,\ 6 > \\ &= < -3+5t,\ 3-4t,\ -2+6t > . \end{aligned}$$

(ii) L intersects the xy-plane when $z(t) = 0 \Rightarrow -2+6t \Rightarrow t = \frac{1}{3}$.
Hence, the point of intersection is $(-\frac{4}{3},\ \frac{5}{3},\ 0)$.

Q5: *(10 points)*

Find the unit tangent and principal unit normal vectors to $r(t) =< \cos(2t), t, \sin(2t) >$ at the point $(1,0,0)$.

Solution:

- The unit tangent vector to $r(t)$ at t is given by:

$$\begin{aligned} T(t) = \frac{r'(t)}{||r'(t)||} &= \frac{1}{\sqrt{4\sin^2(2t)+1+4\cos^2(2t)}} < -2\sin(2t),\ 1,\ 2\cos(2t) > \\ &= \frac{1}{\sqrt{5}} < -2\sin(2t),\ 1,\ 2\cos(2t) > . \end{aligned}$$

  At the point $(1,0,0)$, $t = 0$. Hence $T(0) = \frac{1}{\sqrt{5}} < 0,\ 1,\ 2 > .$

- The principal unit normal vector to $r(t)$ at t is given by:

$$\begin{aligned} N(t) = \frac{T'(t)}{||T'(t)||} &= \frac{1}{\sqrt{16\cos^2(2t)+0+16\sin^2(2t)}} < -4\cos(2t),\ 0,\ -4\sin(2t) > \\ &= \frac{1}{4\sqrt{2}} < -4\cos(2t),\ 0,\ -4\sin(2t) > \end{aligned}$$

  At $t = 0$, $N(0) = \frac{1}{4\sqrt{2}} < -4,\ 0,\ 0 >=< -\frac{1}{\sqrt{2}},\ 0,\ 0 > .$

Q6: *(10 points)*

Sketch the region defined by the limits of the integration

$$\int_0^2 \int_x^{\sqrt{8-x^2}} (x^2+y^2)^{\frac{3}{2}}\, dy\, dx,$$

then evaluate the double integral.

Solution:

The region R of integration is described by $R = \left\{(x,y) |\ 0 \le x \le 2, \quad x \le y \le \sqrt{8-x^2}\right\}$. Converting to polar coordinates gives:

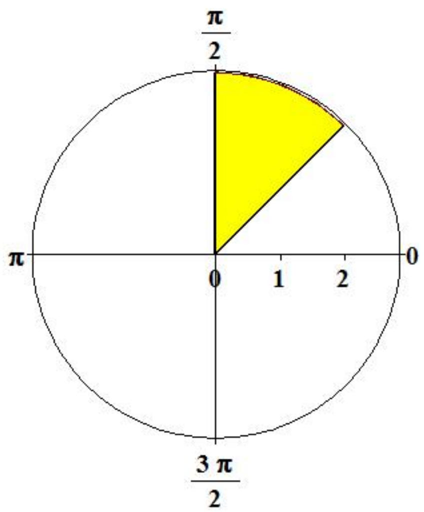

Figure 1: Sketch of the region of integration R

$$\begin{aligned}
\int_0^2 \int_x^{\sqrt{8-x^2}} (x^2+y^2)^{\frac{3}{2}}\, dy\, dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\sqrt{2}} (r^2)^{\frac{3}{2}} r\, dr\, d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\sqrt{2}} r^4\, dr\, d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{r^5}{5}\Big|_0^{2\sqrt{2}}\, dr\, d\theta \\
&= \frac{128\sqrt{2}}{5} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \\
&= \frac{128\sqrt{2}}{5}\, \theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \frac{128\sqrt{2}}{5}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) \\
&= \frac{32\pi\sqrt{2}}{5} \\
&\approx 28.4
\end{aligned}$$

Q7: *(10 points)*

Find the maximum and minimum of the function

$$f(x,y)=x^2y^2 \quad \text{subject to the constraint} \quad x^2+4y^2\le 24.$$

Solution:

$f(x,y)=x^2y^2$ and $R=\{(x,y)|\, x^2+4y^2\le 24\}$

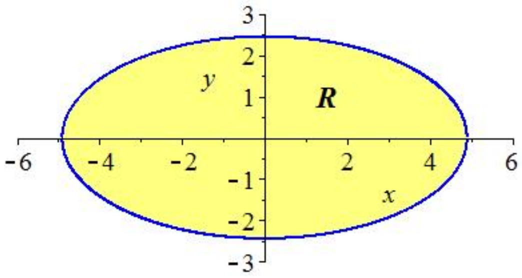

Figure 2: Region R in the xy-plane

$$f_x=2xy^2,\quad f_y=2x^2y.$$

Setting $f_x=0,\ f_y=0$ gives the critical points in R:

$$(0,y),\ -\sqrt{6}\le y\le\sqrt{6}$$

and

$$(x,0),\ -2\sqrt{6}\le x\le 2\sqrt{6}.$$

On the ellipse $x^2+4y^2=24$, $y^2=6-\frac{1}{4}x^2$,

$$f(x,y)=f_1(x)=x^2\left(6-\tfrac{1}{4}x^2\right)=6x^2-\tfrac{1}{4}x^4$$

Differentiating with respect to x gives $f_1'(x)=-x^3+12x$

$$f_1'(x)=0\Rightarrow -x^3+12x=0\Rightarrow -x(x^2-12)=0\Rightarrow x=0,\ x=-2\sqrt{3},\ x=2\sqrt{3}.$$

Since $y=\pm\sqrt{6-\frac{1}{4}x^2}$, we get the 6 points:

$$(x,y)=\left\{(0,-\sqrt{6}),\ (0,\sqrt{6}),\ (2\sqrt{3},-\sqrt{3}),\ (2\sqrt{3},\sqrt{3}),\ (-2\sqrt{3},-\sqrt{3}),\ (2\sqrt{3},\sqrt{3})\right\}$$

Evaluating $f(x,y)$ at these points together with the critical points gives:

$$\begin{array}{ll} f(0,y)=0, & f(x,0)=0 \\ f(0,-\sqrt{6})=0, & f(0,\sqrt{6})=0 \\ f(2\sqrt{3},-\sqrt{3})=36, & f(2\sqrt{3},\sqrt{3})=36 \\ f(-2\sqrt{3},-\sqrt{3})=36, & f(2\sqrt{3},\sqrt{3})=36 \end{array}$$

Hence, $f(x,y)$ has an absolute maximum 36 at the four points:

$$(2\sqrt{3},-\sqrt{3}),\ (2\sqrt{3},\sqrt{3}),\ (-2\sqrt{3},-\sqrt{3}),\ (2\sqrt{3},\sqrt{3})$$

and an absolute minimum 0 along $(0,y)$ and $(x,0)$ for $-2\sqrt{6} \leq x \leq 2\sqrt{6}$ and $-\sqrt{6} \leq y \leq \sqrt{6}$.

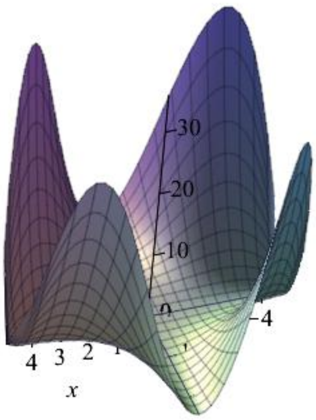

Figure 3: Graph of $f(x,y) = x^2y^2$ in R

Q8: *(8 points)*

Given the transformation

$$x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta) \quad \text{and} \quad z = \rho\cos(\phi).$$

Find the Jacobian

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)},$$

and write your answer in simplest form.

Solution:

The Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}$ is given by: $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \end{vmatrix}$

$$= \begin{vmatrix} \sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{vmatrix}$$

$$= \rho^2\left[\cos^2(\phi)\sin^2(\theta)\sin(\phi) + \cos^2(\phi)\cos^2(\theta)\sin(\phi) + \sin^3(\phi)\sin^2(\theta) + \cos^2(\theta)\sin^3(\phi)\right]$$

$$= \rho^2\sin(\phi)\left[\cos^2(\phi)\sin^2(\theta) + \cos^2(\phi)\cos^2(\theta) + + \sin^2(\phi)\sin^2(\theta) + \cos^2(\theta)\sin^2(\phi)\right]$$

$$= \rho^2\sin(\phi)\left[\cos^2(\phi)\{\sin^2(\theta) + cos^2(\theta)\} + \sin^2(\phi)\{\sin^2(\theta) + \cos^2(\theta)\}\right]$$

$$= \rho^2\sin(\phi)\left[\cos^2(\phi) + \sin^2(\phi)\right] = \rho^2\sin(\phi).$$

Q9: *(8 points)*

Compute the volume of the solid below the surface $z = 1 + e^x \sin y$ and above the region in the xy-plane bounded by the lines $x = -1,\ x = 1,\ y = 0$ and $y = \frac{\pi}{2}$.

Solution:

The region R in the xy-plane is given by $R = \left\{(x, y) \middle| -1 \le x \le 1, \quad 0 \le y \le \frac{\pi}{2}\right\}$.

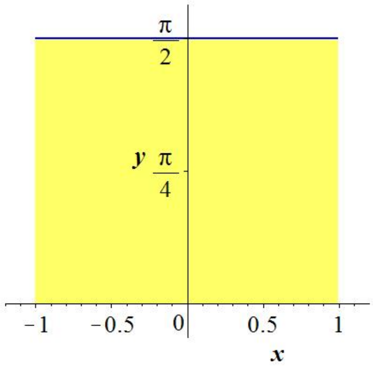

Figure 4: Region R in the xy-plane

Let $z = f(x, y) = z = 1 + e^x \sin y$. Then f is continuous in R and $f(x, y) \ge 0$ in R. Therefore the volume V of the solid below the surface $z = f(x, y)$ and above the region R in the xy-plane is given by:

$$\begin{aligned}
V &= \iint_R f(x,y)dA = \int_{-1}^{1}\int_{0}^{\frac{\pi}{2}} (1 + e^x \sin y)\, dy dx \\
&= \int_{-1}^{1} \left(y - e^x \cos y\right)\Big|_0^{\frac{\pi}{2}}\, dx \\
&= \int_{-1}^{1} \left[(\tfrac{\pi}{2} - 0) - (0 - e^x)\right] dx \\
&= \left(\tfrac{\pi}{2}x + e^x\right)\Big|_{-1}^{1} = \pi + e - e^{-1} \approx 5.49.
\end{aligned}$$

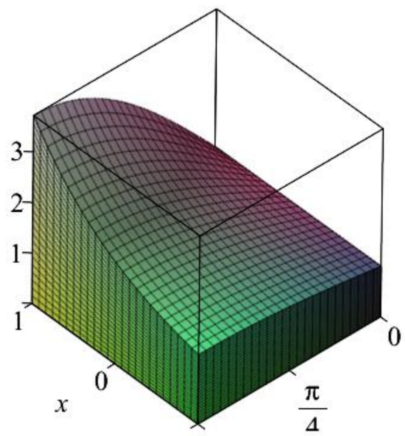

Figure 5: Graph of the solid

Q10: *(10 points)*

Give an accurate statement of Green's theorem. Then use it to evaluate the integral

$$\oint_C (x^3 - y)dx + (x + y^3)dy,$$

where C is the positively oriented curve bounding the region between $y = x^2$ and $y = x$.

Solution:

Green's Theorem:

Let C be a piecewise-smooth, simple closed curve in the plane with positive orientation and let R be the region enclosed by C, together with C. Suppose that $M(x, y)$ and $N(x, y)$ are continuous and have continuous first partial derivatives in some open region D, with $R \subset D$. Then,

$$\oint_C M(x,y)\,dx + N(x,y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dA.$$

The region R of integration is :

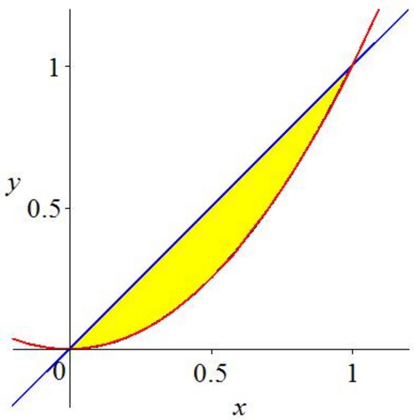

Figure 6: Region of Integration R

Here, we have $M(x, y) = x^3 - y$ and $N(x, y) = y^3 + x$. It follows that:

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - (-1) = 2.$$

Hence, by Green's Theorem

$$\begin{aligned}
\oint_C (x^3 - y)dx + (x + y^3)dy &= \iint_R 2\,dA \\
&= 2\int_0^1 \int_{x^2}^{x} dy\,dx \\
&= 2\int_0^1 y\Big|_{x^2}^{x}\,dx \\
&= 2\int_0^1 (x - x^2)\,dx \\
&= 2\,\left(\tfrac{x^2}{2} - \tfrac{x^3}{3}\right)\Big|_0^1 \\
&= \tfrac{1}{3}.
\end{aligned}$$