Sultan Qaboos University
Department of Mathematics and Statistics

Math2108 | Spring 2011

Test I | Time: 60 minutes

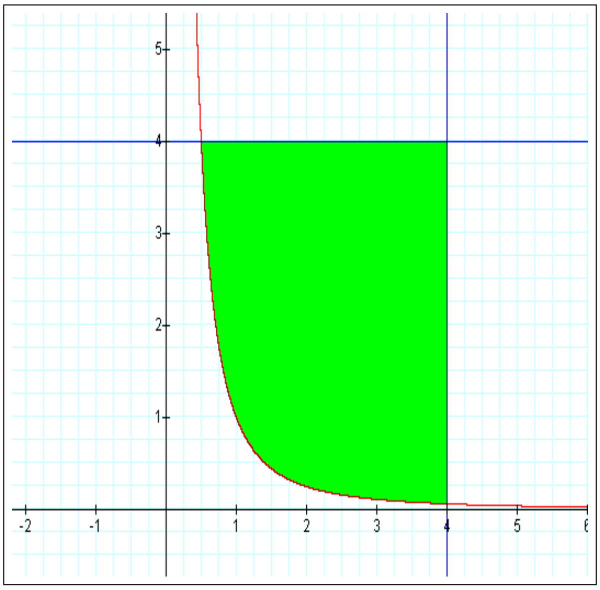
Name:. . . . . SOLUTION: . . . . . . Section: . . . . . Number. . . . .

Important Instructions

- Write your name, ID # and Section # on the front cover of your answer booklet.
- In questions **Q1** to **Q6**, you must show your complete, mathematically correct and neatly written solution. In question **Q7**, circle the choice that best fits the correct answer.
- You are NOT allowed to share calculators or any other material during the test.
- Cellular phones are NOT allowed to be used for any purpose during the test.
- You should NOT ask the invigilator any questions about the exam.

Q1: Find the area of the region bounded by the curves of $y = \frac{1}{x^2}$, $x = 4$ and $y = 4$. *(5 points)*

Solution: First, we find the points of intersection between $y = \frac{1}{x^2}$ and $y = 4$. We solve $\frac{1}{x^2} = 4$ to obtain $x = \pm\frac{1}{2}$. Since the region is in the positive quadrant, then we take $x = \frac{1}{2}$. Now,

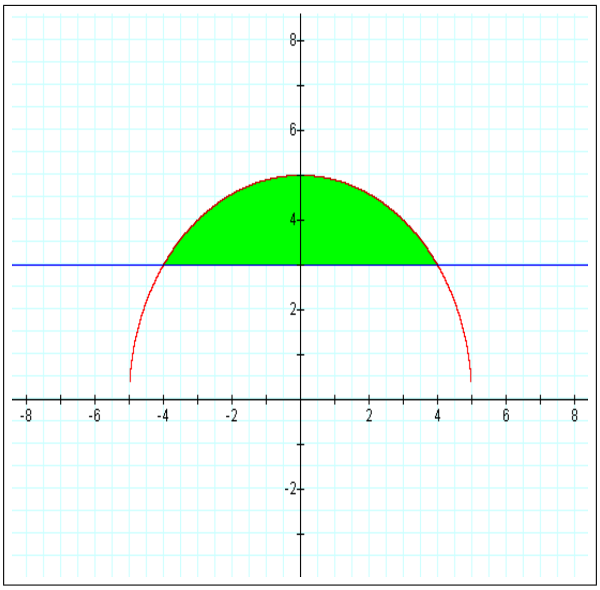


$$\begin{aligned} Area &= \int_{\frac{1}{2}}^{4} (4 - x^{-2})dx \\ &= 4x + \frac{1}{x}\Big]_{\frac{1}{2}}^{4} \\ &= (16 + \frac{1}{4}) - (2 + 2) \\ &= \frac{49}{4}\ units^2 \end{aligned}$$

Q2: The region bounded by the curves of $y = \sqrt{25 - x^2}$ and $y = 3$ is revolved about the x-axis. Sketch the region, and find the volume of the generated solid. *(6 points)*

Solution: We find the points of intersection between $y = \sqrt{25-x^2}$ and $y = 3$ by solving

$$\begin{aligned}\sqrt{25-x^2} =& 3\\ x^2 =& 16\\ x =& \pm 4.\end{aligned}$$



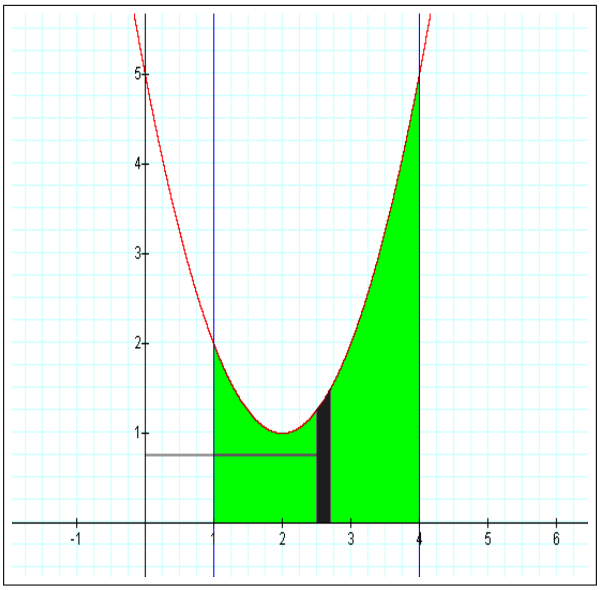
Now, we find the volume using the Washers Method:

$$\begin{aligned}V =& \pi \int_{-4}^{4} (\sqrt{25-x^2})^2 - 3^2 \, dx\\ =& \pi \int_{-4}^{4} (25 - x^2 - 9) \, dx\\ =& \pi \int_{-4}^{4} (16 - x^2) \, dx\\ =& \pi \left[16x - \frac{x^3}{3} \right]_{-4}^{4}\\ =& \frac{256}{3} \pi \; units^3.\end{aligned}$$

Q3: The region bounded by the curves of $y = (x-2)^2 + 1, x = 1, x = 4$ and $y = 0$ is rotated about the y-axis. Sketch the region, and find the volume of the generated solid. *(6 points)*

Solution: We solve the question using the method of Cylindrical Shells:

$$\begin{aligned}V =& 2\pi \int_{1}^{4} x[(x-2)^2 + 1] \, dx\\ =& 2\pi \int_{1}^{4} (x^3 - 4x^2 + 5x) \, dx\\ =& 2\pi \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{5}{2}x^2 \right]_{1}^{4}\\ =& \frac{69}{2} \pi \; units^3.\end{aligned}$$



Q4: Find the surface area of the solid generated by revolving the curve of $y = \sqrt{x}$, $2 \le x \le 4$ about the x-axis. *(6 points)*

`Solution: We apply the formula and do the integral as follows:`

$$
\begin{aligned}
S =& 2\pi \int_2^4 f(x)\sqrt{1 + (f'(x))^2}\, dx \\
=& 2\pi \int_2^4 \sqrt{x}\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx \\
=& 2\pi \int_2^4 \sqrt{x}\frac{\sqrt{4x+1}}{2\sqrt{x}}\, dx \\
=& \pi \int_2^4 \sqrt{4x+1}\, dx \\
=& \pi \left[\frac{2}{3}(4x+1)^{\frac{3}{2}} \cdot \frac{1}{4}\right]_2^4 \\
=& \frac{\pi}{6}(17\sqrt{17} - 27)\ units^2.
\end{aligned}
$$

Q5: Evaluate each of the following integrals. *(6+6 points)*

$$(i)\ \int_0^{\frac{\pi}{4}} \frac{\cos(x)}{\sin^2(x) + 2\sin(x) + 1} dx \qquad\qquad (ii)\ \int \frac{\tan^3(x)}{\sec^2(x)}\, dx.$$

`(i) Solution:`

$$\int_0^{\frac{\pi}{4}} \frac{\cos(x)}{\sin^2(x) + 2\sin(x) + 1} dx = \int_0^{\frac{\pi}{4}} \frac{\cos(x)}{(\sin(x) + 1)^2} dx.$$

`Now, let` $\sin(x) + 1 = y$, `then` $\cos(x)\, dx = dy$ `and the integral becomes`

$$\int_1^{1+\frac{1}{\sqrt{2}}} \frac{1}{y^2}\, dy = \left.\frac{-1}{y}\right]_1^{1+\frac{1}{\sqrt{2}}} = \frac{1}{1+\sqrt{2}}.$$

(ii) Solution: We can solve this question in several ways, here is one of them.

$$\begin{aligned}\int \frac{\tan^3(x)}{\sec^2(x)}\,dx &= \int \frac{\tan(x)\tan^2(x)}{\sec^2(x)}\,dx\\ &= \int \frac{\tan(x)(\sec^2(x)-1)}{\sec^2(x)}dx\\ &= \int \tan(x)dx - \int \frac{\tan(x)}{\sec^2(x)}\,dx\\ &= -\ln|\cos(x)| - \int \frac{\tan(x)\sec(x)}{\sec^3(x)}\,dx.\end{aligned}$$

Now, let $y=\sec(x)$, then $dy=\tan(x)\sec(x)dx$. So, we obtain

$$\int \frac{\tan(x)\sec(x)}{\sec^3(x)}\,dx = \int \frac{1}{y^3}\,dy = \frac{-1}{2}y^{-2}+c = \frac{-1}{2}\sec^{-2}(x)+c.$$

Hence

$$\int \frac{\tan^3(x)}{\sec^2(x)}\,dx = -\ln|\cos(x)| + \frac{1}{2}\sec^{-2}(x)+c.$$

Q6: Let n be a positive integer.

(a) Prove that $\int(\ln x)^n\ dx = x(\ln x)^n - n\int(\ln x)^{n-1}\ dx.$ *(6 points)*

PROOF: Using integration by parts, we can take $u=(\ln x)^n, dv=dx$. In this case, $du=n(\ln x)^{n-1}\frac{1}{x}\ dx$ and $v=x$. Thus,

$$\int (\ln x)^n\ dx = x(\ln x)^n - \int n(\ln x)^{n-1}\frac{1}{x}\ x\ dx = x(\ln x)^n - n\int(\ln x)^{n-1}\ dx.$$

(b) Use the formula in part (a) to evaluate $\int(\ln x)^3\ dx$. *(3 points)*

Solution: We take $n=3$ in part (a) to obtain

$$\int(\ln x)^3\ dx \quad = \quad x(\ln x)^3 - 3\int(\ln x)^2\ dx.$$

Now, use $n=2$ in part (a) to obtain

$$\int(\ln x)^3\ dx \quad = \quad x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\int(\ln x)^1\ dx\right].$$

Finally, use $n=1$ in part (a) to obtain

$$\int(\ln x)^3\ dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + c.$$

Q7 Circle the choice that best fits the correct answer. *(6 points)*

(i) The volume of the solid with cross sectional area $A(x)=\pi(x-1)^2,\ 1\le x\le 7$ is

(A) 63π **(B)** $72\pi^2$ **(C)** $\frac{72}{3}\pi$ **(D)** 72π **(E)** None of these

(ii) The arc length of the curve of $y=(x-1)^{\frac{3}{2}}$ from $x=1$ to $x=6$ is given by

(A) $\int_1^6\sqrt{1+(x-1)^3}dx$ **(B)** $\int_1^6\sqrt{1+\frac{9}{4}(x-1)^3}dx$ **(C)** $\frac{3}{2}\int_1^6\sqrt{x-\frac{5}{9}}dx$ **(D)** None of these

(iii) Suppose that f and g are continuous functions that satisfy $f(x)\ge 0$ and $g(x)\le 0$ for all x. The area between the curves of $f(x)$ and $g(x)$ over the interval [a,b] is

(A) $\int_a^b[f(x)-g(x)]dx$ **(B)** $\int_a^b[g(x)-f(x)]dx$ **(C)** $\int_a^b[f(x)+g(x)]dx$ **(D)** 0

Good Luck