SULTAN QABOOS UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS

Math 2108 Spring 2011
Test II Time: 62 minutes

Name:. . . . . . . Solution . . . . . Section: . . . . . Number. . . . .

Important Instructions

- Write your name, ID # and Section # on the front cover of your answer booklet.
- In questions **Q1** to **Q4**, you must show your complete, mathematically correct and neatly written solution. In question **Q5**, circle the choice that best fits the correct answer.
- You are NOT allowed to share calculators or any other material during the test.
- Cellular phones are NOT allowed to be used for any purpose during the test.
- You should NOT ask the invigilator any questions about the exam.

Q1: (a) Evaluate the integral *(7+5 points)*

$$\int \frac{7x^2+20x+4}{3x(x+1)^2}dx.$$

Solution: We use partial fractions to obtain

$$\begin{aligned}\frac{7x^2+20x+4}{3x(x+1)^2} =& \frac{A}{3x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}.\\ 7x^2+20x+4 =& A(x+1)^2+3Bx(x+1)+3Cx\\ =&(A-7+3B)x^2+(2A+3B+3C-20)x+A-4.\end{aligned}$$

Therefore, $A=4, B=1, C=3$. Now, we integrate to obtain

$$\int \frac{7x^2+20x+4}{3x(x+1)^2}dx = \int\left(\frac{4}{3x}+\frac{1}{x+1}+\frac{3}{(x+1)^2}\right)dx = \frac{4}{3}\ln|x|+\ln|x+1|-\frac{3}{x+1}+c.$$

(b) Determine whether the improper integral

$$\int_0^{\frac{\pi}{4}} \frac{3\sec^2(x)}{\sqrt{1-\tan(x)}}dx.$$

converges or diverges. Find the value of the integral if it converges.

Solution: Since the integral is improper at $x=\frac{\pi}{4}$, then we write

$$\int_0^{\frac{\pi}{4}} \frac{3\sec^2(x)}{\sqrt{1-\tan(x)}}dx = \lim_{R\to\frac{\pi}{4}^-}\int_0^{R} \frac{3\sec^2(x)}{\sqrt{1-\tan(x)}}dx.$$

We use the substitution $y=1-\tan(x)$ to obtain

$$\lim_{R\to\frac{\pi}{4}^-}\int\limits_0^R \frac{3\sec^2(x)}{\sqrt{1-\tan(x)}}dx = \lim_{R\to\frac{\pi}{4}^-}\left(\int\limits_1^{1-\tan(R)} \frac{-3}{\sqrt{y}}dy\right)$$

$$= \lim_{R\to\frac{\pi}{4}^-}\left(-6(\sqrt{1-\tan(R)}-1)\right)$$

$$=6.$$

Q2: Show that the sequence $a_n = \dfrac{e^n}{1+e^n}$ is increasing and bounded. *(5 points)*

`Solution:` `Consider` $f(x) = \frac{e^x}{1+e^x}$, `then`

$$f'(x) = \frac{e^x}{(1+e^x)^2} > 0.$$

`Hence,` a_n `is increasing. Now, since`

$$0 < a_n = \frac{e^n}{1+e^n} < 1$$

`for all values of` n, `then` a_n `is bounded.`

Q3: Test whether each one of the following series is convergent or divergent: *(6+5+5 points)*

$$(a)\ \sum_{k=1}^{\infty}\frac{2-\tan^{-1}(k)}{k^2+1} \qquad (b)\ \sum_{k=1}^{\infty}\frac{\sqrt{k^3+2k+3}}{k^3+5k-2} \qquad (c)\ \sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k\ln(k)}.$$

`Solution:`

`(a) Since` $\frac{-\pi}{2} < -\tan^{-1}(k) < \frac{\pi}{2}$, `then`

$$2-\frac{\pi}{2} < 2-\tan^{-1}(k) < 2+\frac{\pi}{2}$$

`and`

$$0 < \frac{2-\frac{\pi}{2}}{1+k^2} < \frac{2-\tan^{-1}(k)}{1+k^2} < \frac{2+\frac{\pi}{2}}{1+k^2} < (2+\frac{\pi}{2})\frac{1}{k^2}.$$

`Since` $\sum_{k=1}^{\infty}\frac{1}{k^2}$ `is a convergent` p`-series, then the series in (a) is convergent by the comparison test.`

`(b) Take`

$$a_k = \frac{\sqrt{k^3+2k+3}}{k^3+5k-2}, \qquad b_k = \frac{1}{k^{\frac{3}{2}}}$$

`and use the limit comparison test. We have`

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{\sqrt{1+2\frac{1}{k^2}+\frac{3}{k^3}}}{1+5\frac{1}{k^2}-\frac{2}{k^3}} = 1.$$

Since $\sum_{k=1}^{\infty} b_k$ is a convergent p-series, then the series in (b) is convergent by the limit comparison test.

(c) This series is alternating. Since

$$(i)\ \frac{1}{k\ln(k)} > 0 \quad \texttt{for all } k \geq 2$$

$$(ii)\ \lim_{k\to\infty} \frac{1}{k\ln(k)} = 0$$

and

$$(iii)\ \frac{\frac{1}{(k+1)\ln(k+1)}}{\frac{1}{k\ln(k)}} = \frac{k}{k+1}\frac{\ln(k)}{\ln(k+1)} < 1$$

for all $k > 1$, then we use the alternating series test to conclude that the series in (c) is convergent.

Q4: Determine whether the series *(7 points)*

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3^k}{k\ 4^k}$$

is absolutely convergent, conditionally convergent or divergent.

Solution: Since

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\frac{3^{k+1}}{(k+1)4^{k+1}}}{\frac{3^k}{k4^k}}\right| = \lim_{k\to\infty}\frac{3k}{4(k+1)} = \frac{3}{4} < 1,$$

then the series is absolutely convergent by the ratio test.

Since the series is absolutely convergent, then it cannot be conditionally convergent, and it cannot be divergent.

Name:....... Solution .... Section: ..... Number........

You must keep this page stapled with your answer booklet

Q5: Circle the choice that best fits the correct answer. *(2 points each)*

- One of the following sequences is divergent

  (**A**) $a_n = \frac{1}{n}$ [✓ (**B**) $b_n = (-1)^n \frac{2n^3}{n^3+1}$]

  (**C**) $c_n = \frac{(-1)^n}{n}$ (**D**) $S_n = \sum_{k=1}^{n} \left(\frac{-1}{5}\right)^k$

- One of the following series is conditionally convergent:

  (**A**) $\sum_{k=1}^{\infty} \frac{1}{k}$ (**B**) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$

  [✓ (**C**) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$] (**D**) $\sum_{k=1}^{\infty} (-1)^k$

- One of the following is true:

  (**A**) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{k=1}^{\infty} a_k$ is convergent.
  (**B**) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{k=1}^{\infty} a_k$ is divergent.
  (**C**) [✓ If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$,]
  (**D**) If $\sum_{k=1}^{\infty} a_k$ is divergent, then $\lim a_k \neq 0$.

- A student needed to test the series $\sum_{k=1}^{\infty} \dfrac{1}{k + |\sin(k)|}$ for convergence/divergence. The student solved the problem as follows:

  `Step 1:` $0 < \dfrac{1}{k + |\sin(k)|} \leq \dfrac{1}{k}$ for all $k \geq 1$.

  `Step 2:` We know that $\sum_{k=1}^{\infty} \dfrac{1}{k}$ is a divergent p-series with $p = 1$.

  `Step 3:` Because $\sum_{k=1}^{\infty} \dfrac{1}{k}$ is divergent, then $\sum_{k=1}^{\infty} \dfrac{1}{k + |\sin(k)|}$ is divergent by the comparison test.

  What do you think about the solution of this student?

  (**A**) The solution is correct (**B**) Step 1 is wrong
  (**C**) Step 2 is wrong [✓ (**D**) Step 3 is wrong]

- One of the following integrals is improper:

  (**A**) $\int_0^1 \dfrac{\cos(x)}{1 + \sin(x)} dx$ (**B**) $\int_0^1 \dfrac{\ln(x+1)}{1 + \ln(x+1)} dx$

  [✓ (**C**) $\int_0^1 \dfrac{\cos(x)}{1 + \cos(\pi x)} dx$] (**D**) $\int_0^1 \dfrac{\sin(x)}{1 + \cos(x)} dx$

Good Luck