Sultan Qaboos University
Department of Mathematics and Statistics

Math 2108 — Spring 2011

Final Exam — Time: 2 hours & 30 minutes

Name: . . . . . .Solution. . . . . . . Section: . . . . . ID Number. . . . .

Important Instructions

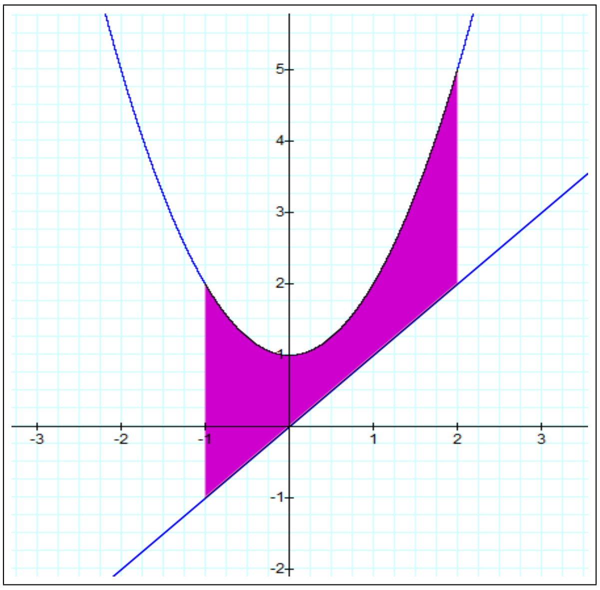
- Write your name, ID # and Section # on the front cover of your answer booklet.
- In questions **Q1** to **Q8**, you must show your complete, mathematically correct and neatly written solution. In question **Q9**, circle the choice that best fits the correct answer.
- You are NOT allowed to share calculators or any other material during the test.
- Cellular phones are NOT allowed to be used for any purpose during the test.
- You should NOT ask the invigilator any questions about the exam.

Total Points: 100

Q1: *(6+6 points)*

(a) Sketch the region enclosed by $y = x^2 + 1, y = x, x = -1$ and $x = 2$, then find its area.

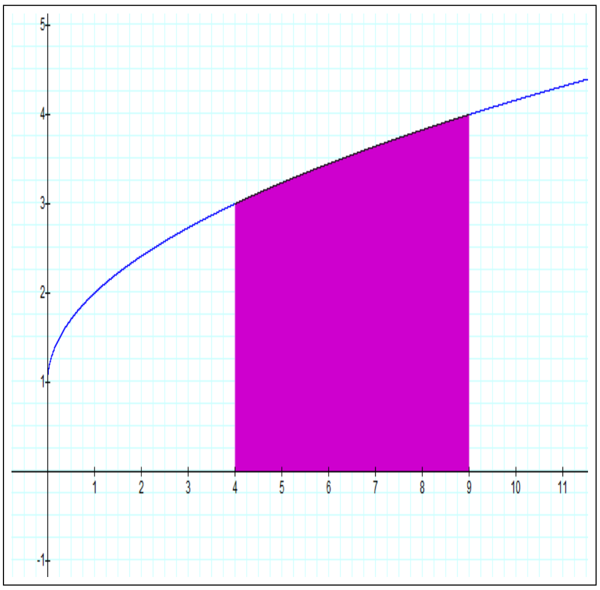
Solution:



$$A = \int_{-1}^{2} (x^2 + 1 - x)\, dx$$
$$= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + x\right]_{-1}^{2}$$
$$= \frac{9}{2}.$$

(b) Sketch the region bounded by the curves of $y = 1 + \sqrt{x}, x = 4, x = 9$ and $y = 0$, then find the volume of the solid generated by revolving the region about the y-axis.

`Solution:`



Using the method of cylindrical shells, we obtain

$$\begin{aligned} V =& 2\pi \int_4^9 x(1 + \sqrt{x})\, dx \\ =& 2\pi \left[\frac{1}{2}x^2 + \frac{2}{5}x^{\frac{5}{2}} \right]_4^9 \\ =& \frac{1169}{5}\pi. \end{aligned}$$

Q2: Evaluate each of the following integrals: *(6+6 points)*

$$(a)\ \int \frac{1}{x^2}\sec\left(\frac{1}{x}\right)dx \qquad\qquad (b)\ \int \sqrt{9-x^2}dx$$

`Solution (a)` Let $y=\frac{1}{x}$, then $-x^2dy=dx$. Therefore,

$$\begin{aligned}\int \frac{1}{x^2}\sec\left(\frac{1}{x}\right)dx &= -\int \sec(y)dy\\ &= -\int \sec(y)\frac{\sec(y)+\tan(y)}{\sec(y)+\tan(y)}dy\\ &= -\int \frac{\sec^2(y)+\sec(y)\tan(y)}{\sec(y)+\tan(y)}dy\\ &= -\ln|\sec(y)+\tan(y)|+c\\ &= -\ln\left|\sec\left(\frac{1}{x}\right)+\tan\left(\frac{1}{x}\right)\right|+c\end{aligned}$$

`Solution (b)` Let $x=3\sin(y)$, then $dx=3\cos(y)dy$. Therefore,

$$\begin{aligned}\int \sqrt{9-x^2}dx &= \int 3\cos(y)\sqrt{9-9\sin^2(y)}dy\\ &= \int 9\cos(y)\sqrt{1-\sin^2(y)}dy\\ &= \int 9\cos^2(y)dy\\ &= 9\int \frac{1+\cos(2y)}{2}dy\\ &= \frac{9}{2}y+\frac{9}{4}\sin(2y)+c\\ &= \frac{9}{2}\sin^{-1}\left(\frac{1}{3}x\right)+\frac{1}{2}x\sqrt{9-x^2}+c\end{aligned}$$

Q3: Evaluate each of the following integrals: *(6+6 points)*

$$(a)\ \int_0^{\pi} \sin^4(x)dx \qquad\qquad (b)\ \int_1^{\infty} \frac{e^{-x}}{\sqrt{1-e^{-x}}}dx$$

`Solution (a)` We use integration by parts to obtain

$$\begin{aligned}\int_0^{\pi} \sin^4(x)\ dx =& \int_0^{\pi} \sin(x)\sin^3(x)\ dx \\ =& -\sin^3(x)\cos(x)]_0^{\pi} + 3\int_0^{\pi} \sin^2(x)\cos^2(x)\ dx \\ =& 0 + 3\int_0^{\pi} \sin^2(x)(1-\sin^2(x))\ dx\end{aligned}$$

Thus,

$$\int_0^{\pi} \sin^4(x)\ dx = \frac{3}{4}\int_0^{\pi} \sin^2(x)\ dx = \frac{3}{8}\int_0^{\pi} (1-\cos(2x))\ dx.$$

Now, we use a trig identity to obtain

$$\begin{aligned}\int_0^{\pi} \sin^4(x)\ dx =& \frac{3}{8}x]_0^{\pi} - \frac{3}{16}\sin(2x)]_0^{\pi} \\ =& \frac{3\pi}{8}.\end{aligned}$$

`Solution (b)` This integral is an improper integral. Let $1-e^{-x}=y$, then $e^{-x}dx = dy$, and therefore,

$$\begin{aligned}\int_1^{\infty} \frac{e^{-x}}{\sqrt{1-e^{-x}}}dx =& \lim_{R\to\infty} \int_1^{R} \frac{e^{-x}}{\sqrt{1-e^{-x}}}dx \\ =& \lim_{R\to\infty} \int_{1-e^{-1}}^{1-e^{-R}} \frac{1}{\sqrt{y}}dy \\ =& \lim_{R\to\infty} [2\sqrt{y}]_{1-e^{-1}}^{1-e^{-R}} \\ =& \lim_{R\to\infty} \left(2\sqrt{1-e^{-R}} - 2\sqrt{1-e^{-1}}\right) \\ =& 2(1-\sqrt{1-e^{-1}}).\end{aligned}$$

Q4: Determine whether each of the following series is conditionally convergent, absolutely convergent or divergent. *(5+7 points)*

$$(a) \sum_{k=0}^{\infty} \frac{(-1)^{k+1} k!}{3^k} \qquad (b) \sum_{k=1}^{\infty} \frac{(-1)^k 2k}{k^2+1}.$$

`Solution (a)` We use the k-th term test. Since

$$\frac{k!}{3^k} = \frac{k}{3} \frac{(k-1)}{3} \frac{(k-2)}{3} \cdots \frac{4}{3}\frac{3}{3}\frac{2}{3}\frac{1}{3} > \frac{2}{9},$$

for all $k > 3$, then

$$\lim_{k\to\infty} \frac{(-1)^{k+1} k!}{3^k}$$

does not exist, and the series is divergent by the kth term test.

`Solution (b)` First, we test for absolute convergence. The series $\sum_{k=1}^{\infty} \frac{2k}{k^2+1}$ is divergent by the limit comparison test with $b_k = \frac{1}{k}$, i.e., since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2}{n^2+1} = 2$$

and since $\sum_{k=1}^{\infty} b_k$ is divergent p-series, then $\sum_{k=1}^{\infty} \frac{2k}{k^2+1}$ is divergent too.

Next, we test for conditional convergence. Using the alternating series test, we take $a_n = \frac{2n}{n^2+1}$. a_n is positive and $\lim_{n\to\infty} a_n = 0$. Also, since $f(x) = \frac{2x}{x^2+1}$ implies

$$f'(x) = \frac{-2(x-1)(x+1)}{(x^2+1)^2} < 0 \quad \text{for all} x > 1,$$

then a_n is decreasing. Hence, the series in (a) is convergence by the alternating series test. From these facts, we have conditional convergence.

Q5: Consider $f(x) = \ln(x)$ and answer each of the following: *(7+2 points)*

(a) Find the Taylor series of $f(x)$ about $x = 1$, and find the interval of convergence.

`Solution:` Since

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x}, \quad f''(x) = \frac{-1}{x^2}, \ldots, f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$$

then

$$f(1) = 0, \quad f'(1) = 1, \quad f''(1) = -1, \ldots, f^{(n)}(1) = (-1)^{n+1}(n-1)!,$$

and therefore, the Taylor series about $x = 1$ is given by

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \ldots + (-1)^{n+1}\frac{1}{n}(x-1)^n + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k}.$$

From the ratio test,

$$\lim_{k\to\infty} \left| \frac{(x-1)^{k+1}k}{(k+1)(x-1)^k} \right| = |x-1| \lim_{k\to\infty} \frac{k}{k+1} = |x-1|$$

and $|x-1| < 1$ implies $0 < x < 2$. At the boundary, we have

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k}{k} = \sum_{k=1}^{\infty} \frac{-1}{k} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}.$$

Thus, 0 is not included and 2 is included. The interval of convergence becomes $(0, 2]$.

(b) Can we use the series obtained in part (a) to represent $\ln(5)$? Justify your answer.

`Answer 1:` No because 5 is out of the interval of convergence.

`Answer 2:` Yes because $\ln(5) = -\ln(\frac{1}{5})$ and we can use $\frac{1}{5}$ in the series given in part (a).

Q6: Use the geometric series representation of $g(x) = \dfrac{1}{1-x}$, $x \in (-1, 1)$ to find the Maclaurin series of *(5+5 points)*

$$(a)\ \frac{1}{1+x^2} \qquad\qquad (b)\ \tan^{-1}(x).$$

`Solution:` since

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots x^n + \ldots$$

then

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 + \ldots + (-1)^n x^{2n} + \ldots$$

and

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}\, dx = c + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \ldots + \frac{(-1)^n}{2n+1}x^{2n+1} + \ldots$$

Since we know that $\tan^{-1}(0) = 0$, then the constant of integration must be $c = 0$. Hence, the Maclaurin series of $\tan^{-1}(x)$ is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

Q7: Find the interval and radius of convergence for each of the following: *(6+6 points)*

$$(a)\ \sum_{k=1}^{\infty} \frac{(2x+1)^k}{(k+1)^2} \qquad\qquad (b)\ \sum_{k=1}^{\infty} \left(\frac{4k}{5k+1}\right)^k x^k$$

`Solution (a)` We use the ratio test to obtain

$$\lim_{k\to\infty} \left| \frac{(2x+1)^{k+1}(k+1)^2}{(k+2)^2(2x+1)^k} \right| = |2x+1| \lim_{k\to\infty} |2x+1| \frac{(k+1)^2}{(k+2)^2} = |2x+1|.$$

$|2x+1| < 1$ if and only if $-1 < x < 0$. Now, we test the boundary.

At $x = 0$, we have

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)^2} = \sum_{k=2}^{\infty} \frac{1}{k^2},$$

which is a convergent p-series.

At $x = -1$, we have

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)^2},$$

which is convergent absolutely as shown when $x = 0$.

Hence, then interval of convergence is $[-1, 0]$ and the radius of convergence is $\frac{1}{2}$.

`Solution (b)` We use the root test to obtain

$$\lim_{k\to\infty} \left| \left(\frac{4k}{5k+1}\right)^k x^k \right|^{\frac{1}{k}} = \lim_{k\to\infty} |x| \frac{4k}{5k+1} = \frac{4}{5}|x|.$$

$\frac{4}{5}|x| < 1$ if and only if $\frac{-5}{4} < x < \frac{5}{4}$. Next, we need to test the boundary.

At $x = \frac{5}{4}$, we obtain

$$\sum_{k=1}^{\infty} \left(\frac{4k}{5k+1}\right)^k \left(\frac{5}{4}\right)^k = \sum_{k=1}^{\infty} \left(\frac{20k}{20k+4}\right)^k = \sum_{k=1}^{\infty} \left(1 - \frac{4}{20k+4}\right)^k.$$

Next, we use the kth term test to obtain

$$\lim_{k\to\infty} \left(1 - \frac{4}{20k+4}\right)^k = \lim_{n\to\infty} \left(1 - \frac{4}{n}\right)^{\frac{n}{20} - \frac{1}{5}} = \lim_{n\to\infty} \frac{\left((1-\frac{4}{n})^n\right)^{\frac{1}{20}}}{(1-\frac{4}{n})^{\frac{1}{5}}} = e^{\frac{-1}{5}} \neq 0.$$

At $x = \frac{-5}{4}$, we obtain

$$\sum_{k=1}^{\infty} \left(\frac{4k}{5k+1}\right)^k \left(\frac{-5}{4}\right)^k = \sum_{k=1}^{\infty} (-1)^k \left(\frac{20k}{20k+4}\right)^k = \sum_{k=1}^{\infty} (-1)^k \left(1 - \frac{4}{20k+4}\right)^k.$$

Again here, we use the previous facts to conclude that the limit of the k-th term does not exist.

Hence, the interval of convergence is $(\frac{-5}{4}, \frac{5}{4})$ and the radius of convergence is $\frac{5}{4}$.

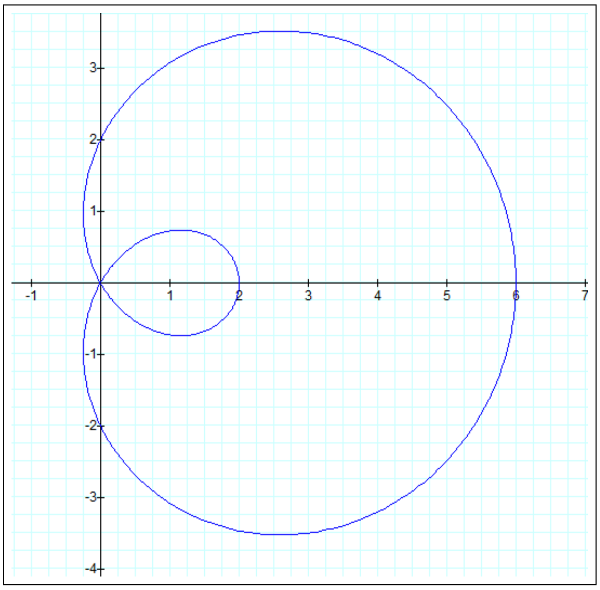
Q8: *(3+6 points)*

(a) Find a polar equation corresponding to the rectangular equation $x^2 + y^2 = x$.

`Solution:` Since $x^2 + y^2 = r^2$ and $x = r\cos(\theta)$, then the polar equation is

$$r^2 = r\cos(\theta) \qquad \text{or} \qquad r = \cos(\theta).$$

(b) Sketch the graph of the polar equation $r = 2 + 4\cos(\theta)$.



with proper justification like finding places where $r = 0$ and taking a table of points.

θ	0	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$
r	6	2	0	-2	0	2	4

Q9: Circle the choice that best fits the correct answer. *(2 points each)*

- The arc length of the curve of $x = \sin(y^2)$ from $y = 0$ to $y = \pi$ is given by

  (**A**) $\int_0^\pi \sqrt{1 + 4x^2 \cos(x)}dx$ (**B**) $\int_0^\pi 2\pi y\sqrt{1 + 4y^2 \sin(y)}dy$

  $\boxed{(\mathbf{C})}$ $\int_0^\pi \sqrt{1 + 4y^2 \cos^2(y^2)}dy$ (**D**) None of the above.

- The series $\sum_{k=0}^{\infty} \frac{3^k + 4(-2)^k}{6^k}$ is

  (**A**) convergent to zero (**B**) divergent

  $\boxed{(\mathbf{C})}$ convergent to 5 (**D**) none of the above.

- One of the following is a non-power series

  (**A**) $\sum_{k=1}^{\infty} x^k$ (**B**) $\sum_{k=1}^{\infty} k^{\frac{1}{k}}(x-1)^k$

  (**C**) $\sum_{k=1}^{\infty} \sin(k)(x+5)^k$ $\boxed{(\mathbf{D})}$ $\sum_{k=1}^{\infty} 3k! \cos(x^k)$

- The polar point $(r, \theta) = (1, \frac{\pi}{2})$ has the following rectangular representation

  (**A**) $(1, \frac{5\pi}{2})$ $\boxed{(\mathbf{B})\ (0, 1)}$ (**C**) $(1, \frac{-3\pi}{2})$ (**D**) none of these.

- The rectangular point $(x, y) = (1, -1)$ has the following polar representation

  (**A**) $(-\sqrt{2}, \frac{-\pi}{4})$ (**B**) $(\sqrt{2}, \frac{\pi}{4})$ $\boxed{(\mathbf{C})\ (\sqrt{2}, \frac{-\pi}{4})}$ (**D**) none of these.

- One of the following sequences is convergent

  (**A**) $a_n = (-1)^n$ (**B**) $b_n = \sin(n)$ $\boxed{(\mathbf{C})\ c_n = \sqrt{n} - \sqrt{n+1}}$ (**D**) $s_n = \cos(n)$

Good Luck