Periodic Orbits in Periodic Discrete Dynamics

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Abstract

We study the combinatorial structure of periodic orbits of nonautonomous difference equations $x_{n+1} = f_n(x_n)$ in a periodically fluctuating environment. We define the Γ-set to be the set of minimal periods that are not multiples of the phase period. We show that when the functions f_n are rational functions, the Γ-set is a finite set. In particular, we investigate several mathematical models of single-species without age structure and find that periodic oscillations are influenced by periodic environments to the extent that almost all periods are divisors or multiples of the phase period.

Key words: Periodic difference equations; periodic orbits; combinatorial dynamics; population models.

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1 Introduction

Given a continuous function $f : I \to I$, where I is a closed interval (I can be $\mathbb{R}$), the orbit of the autonomous difference equation

$$x_{n+1} = f(x_n) \tag{1.1}$$

through a point $x_0 \in I$ is defined as

$$\mathcal{O}^+(x_0) := \{x_0, \overbrace{f(x_0)}^{x_1}, \overbrace{f^2(x_0)}^{x_2}, \overbrace{f^3(x_0)}^{x_3}, \ldots\}, \tag{1.2}$$

where we use the symbol f^n to denote $f \circ \cdots \circ f \circ f$ (n times). We call such orbit an autonomous orbit; it is called periodic of minimal period r (or an r-cycle)

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if the sequence $x_0, x_1, x_2, \ldots$ is periodic, and r is the smallest positive integer for which $x_{n+r} = x_n$, for all $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$. In 1965, Sharkovsky [36] proved that the existence of an r-cycle of equation (1.1) assures the existence of k-cycles for all $r \prec k$ in the following ordering.

$$3 \prec 5 \prec \cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec \cdots 2^n \cdot 3 \prec 2^n \cdot 5 \prec \cdots \prec 2^n \prec \cdots \prec 2^2 \prec 2 \prec 1.$$

For instance, if $f(x) = \frac{-3}{2}x^2 + \frac{5}{2}x + 1$ in Eq. (1.1), then $\{0, 1, 2\}$ is a 3-cycle. By Sharkovsky's theorem, Eq. (1.1) has k-cycles for all $k \in \mathbb{Z}^+ := \mathbb{N} \setminus \{0\}$.

Since autonomous equations (1.1) do not account for fluctuating environments in mathematical modelling, there have been several contributions concerned with nonautonomous periodic difference equations

$$x_{n+1} = f(n \bmod p, x_n) = f_{n \bmod p}(x_n),\ p > 1,\ n \in \mathbb{N}. \tag{1.3}$$

See for instance AlSharawi [1], AlSharawi and Angelos [2], AlSharawi et al. [3], Clark and Gross [6], Cushing and Henson [9,10], Elaydi and Sacker [11,12], Franke and Selgrade [14], Franke and Yakubo [15–19], Henson [23], Kon [26,27], Kot [28]. Here, we consider the p-periodic sequence of functions in Eq. (1.3) to be

$$\{f_0, f_1, f_2, \cdots, f_{p-1}\} \subset \mathcal{C}(I), \tag{1.4}$$

where $\mathcal{C}(I)$ is the space of continuous functions on a closed interval I. Thus, p is the smallest positive integer for which $f_{p+n} = f_p,\ \forall n \in \mathbb{N}$. In this case, we call Eq. (1.3) a p-periodic difference equation. The nonautonomous orbit of the p-periodic difference equation (1.3) through a point $x_0 \in I$ is defined as $\mathcal{O}^+(x_0) :=$

$$\{x_0, \overbrace{f_0(x_0)}^{x_1}, \ldots, \overbrace{f_{p-1} \cdots f_0(x_0)}^{x_p}, \overbrace{f_0 f_{p-1} \cdots f_0(x_0)}^{x_{p+1}}, \overbrace{f_1 f_0 f_{p-1} \cdots f_0(x_0)}^{x_{p+2}}, \ldots\}. \tag{1.5}$$

If r is the smallest positive integer for which $x_{n+r} = x_n, \forall n \in \mathbb{N}$, then the orbit in (1.5) is called periodic of minimal period r. Elaydi and Sacker [11,12] introduced the notion of geometric cycle to distinguish the periodic orbit in (1.5) from the autonomous periodic orbit that results from a skew product semi-dynamical system associated with (1.5). Here, we adopt the same notation and call it a geometric r-cycle (or r-cycle for short).

In 2006, AlSharawi et al. [3] extended Sharkovsky's ordering of the positive integers to what they call "p-Sharkovsky's ordering"

$$
\begin{aligned}
&\mathcal{A}_{p,3} \prec \mathcal{A}_{p,5} \prec \mathcal{A}_{p,7} \prec \ldots \\
&\mathcal{A}_{p,2\cdot 3} \prec \mathcal{A}_{p,2\cdot 5} \prec \mathcal{A}_{p,2\cdot 7} \prec \ldots \\
&\vdots \\
&\mathcal{A}_{p,2^n\cdot 3} \prec \mathcal{A}_{p,2^n\cdot 5} \prec \mathcal{A}_{p,2^n\cdot 7} \prec \ldots \\
&\vdots \\
&\cdots \prec \mathcal{A}_{p,2^n} \prec \ldots \prec \mathcal{A}_{p,2^2} \prec \mathcal{A}_{p,2} \prec \mathcal{A}_{p,1},
\end{aligned}
\tag{1.6}
$$

where $\mathcal{A}_{p,q} := \{m \in \mathbb{N} : \mathrm{lcm}(m,p) = pq\}$. Moreover, they proved, among other things, that if the p-periodic difference equation in (1.3) has a geometric r-cycle, then each set $\mathcal{A}_{p,q}$, $\mathcal{A}_{p,r} \prec \mathcal{A}_{p,q}$, contains at least one period of a geometric cycle. In general, it is unknown which elements of $\mathcal{A}_{p,q}$ are minimal periods and which are not. Canovas and Linero [5] attempted to give a refinement of this result, particularly, for $p = 2$; however, $p = 2$ is a very special case since any positive integer is either a multiple of p or relatively prime with p.

Two main notions proved to be of particular interest in periodic discrete dynamical systems. The first is concerned with whether a periodic environment is advantageous (the average densities are greater in a periodic environment than in a constant environment) [7,11,12,24] or deleterious (the average densities are less in a periodic environment than in a constant environment) [15,16,23,26,27]. Jillson [25] in an experimental study has shown that environment is advantageous, and Henson and Cushing [24] provided integrated theoretical and experimental results to illustrate the possibility of increased population numbers in a periodically fluctuating environment; however, theoretical studies prove that this notion is model dependent [15,16,35]. The second notion is concerned with global stability [9,11,12], which shows that the period of a globally asymptotically stable cycle must divide the phase period in a connected metric space. The main objective of this paper is to emphasize a third notion, which is periodic oscillations are influenced by the periodic environment to the extent that -even in models that portray complicated dynamics- "almost all" periods are divisors or multiples of the phase period. For easy reference, let us define what we call the Γ-set of Eq. (1.3).

Definition 1.1 The Γ-set of Eq. (1.3) is the set of positive integers r such that Eq. (1.3) has a geometric r-cycle and r is not a multiple of p, i.e., $r \notin \{p, 2p, 3p, \ldots\}$.

In this paper, we focus on the combinatorial structure of geometric cycles and use it to investigate the cardinality of the Γ-set. In Section 2, we make a comparison between the combinatorial structure of autonomous and nonautonomous orbits. In Section 3, we investigate the size of the Γ-set when

$f_i,\ i = 0 \cdots p-1$ are polynomials and we extend the results to rational maps in Section 4. In Section 5, we apply the developed ideas to several mathematical models. Finally, we discuss the structured stability of the orbits of Eq. (1.5) in Section 6.

2 Autonomous versus nonautonomous

To give a better understanding of the combinatorial structure of geometric cycles, we make a comparison between cycles of Eq. (1.1) and geometric cycles of Eq. (1.3). In particular, it is well-known that for autonomous orbits

(i) The elements of an r-cycle must be distinct.
(ii) Different cycles are disjoint.
(iii) If $\{x_0, x_1, \ldots, x_{r-1}\}$ is an r-cycle, then any phase shift

$$\{x_j, x_{j+1 \bmod r}, \ldots, x_{r-1+j \bmod r}\},\ 0 \leq j \leq r-1$$

represents the same cycle.

Before making a contrast, let us give the following example:

Example 2.1 *Consider the* 6*-periodic sequence* $\{f_0, f_1, \cdots, f_5\}$, *where*

$$\begin{aligned} f_0(x) &= x+1 & f_3(x) &= x^2-5x+6 \\ f_1(x) &= -x+3 & f_4(x) &= -x^2+3x+1 \\ f_2(x) &= x^2-x+1 & f_5(x) &= -x^2+3x. \end{aligned}$$

$\{0,1,2,3\}$ *is a geometric* 4*-cycle. It is straightforward to check that* $\{2,3,0,1\}$ *represents the same geometric* 4*-cycle; however,* $\{1,2,3,0\}$ *is not a geometric cycle. To emphasize another aspect of periodic orbits, let us define the functions* $\{f_0, f_1, \cdots, f_5\}$ *as follows:*

$$\begin{aligned} f_0(x) &= x & f_3(x) &= (1-x)^2 \\ f_1(x) &= 1-x & f_4(x) &= x^3 \\ f_2(x) &= x^2 & f_5(x) &= (1-x)^3. \end{aligned}$$

Then $\{0,0,1,1\}$ *is a geometric* 4*-cycle, which represents the same geometric cycle as* $\{1,1,0,0\}$.

Now, it is evident that geometric cycles are ordered sets. To make a comparison with cycles of Eq. (1.2), we give ourselves the liberty to neglect the order, and hence, we can state the following:

(i) The elements of a geometric cycle are not necessarily distinct.
(ii) Different geometric cycles are not necessarily disjoint.
(iii) If $\{x_0, x_1, \cdots, x_{r-1}\}$ is a geometric r-cycle, then its phase shifts are given by

$$\{x_{jd}, x_{jd+1 \bmod r}, x_{jd+2 \bmod r}, \cdots, x_{jd-1}\},$$

where $0 \leq j \leq \frac{r}{d} - 1$ and d represents the greatest common divisor between r and p ($d := \gcd(r,p)$).

Consider the p-periodic sequence of functions in (1.4), and let $\{x_0, x_1, \cdots, x_{r-1}\}$ be a geometric r-cycle. Although the notion of skew product semi-dynamical system used in [11,12] can be used to clarify our discussion here, we prefer to clarify the discussed notions by putting an algebraic structure on geometric cycles, this notion can be extracted from [1]. Write the orbit of this geometric cycle in matrix form as follows:

$$\begin{array}{ccccccccc} & f_0 & & f_1 & & f_2 \; \cdots & f_{p-2} & & f_{p-1} \\ \hline x_0 & \to & x_1 & \to & x_2 & \to \cdots & \to & x_{p-1} & \to \\ x_p & \to & x_{p+1} & \to & x_{p+2} & \to \cdots & \to & x_{2p-1} & \to \\ x_{2p} & \to & x_{2p+1} & \to & x_{2p+2} & \to \cdots & \to & x_{3p-1} & \to \\ \vdots & \to & \vdots & \to & \vdots & \to \cdots & \to & \vdots & \to \\ x_{(\frac{l}{p}-1)p} & \to & x_{(\frac{l}{p}-1)p+1} & \to & x_{(\frac{l}{p}-1)p+2} & \to \cdots & \to & x_{l-1} & \end{array} \tag{2.1}$$

where $l = \text{lcm}(r,p)$ denotes the least common multiple between r and p, and the x-indices are carried modulo r. Obviously, if we continue the orbit, then the same pattern repeats. Define the operation $\star$ on the set $G := \{x_0, x_1, \cdots, x_{r-1}\}$ as follows: $x_i \star x_j = x_{i+j \bmod r}$, $0 \leq i, j \leq r-1$. $(G, \star)$ is a group isomorphic to $(Z_r, +)$. Here we must alert the reader to the fact that the set G is an ordered set, for instance, $G = \{1,1,2\} \neq \{1,2\}$. Now, define $G_0 := \{x_0, x_p, \cdots, x_{(\frac{l}{p}-1)p}\}$, then $(G_0, \star)$ is a cyclic subgroup of $(G, \star)$. Consequently, $G_i := x_i \star G_0$, $1 \leq i \leq d-1$ define the remaining $d-1$ cosets. Observe that the map f_i, $1 \leq i \leq p-1$ maps the coset $G_{i \bmod d}$ onto the coset $G_{i+1 \bmod d}$. Following this discussion and using Lagrange's theorem, the next proposition is straightforward.

Proposition 2.2 *Consider the p-periodic sequence in (1.4) and let $d = gcd(r,p)$. Each of the following statements holds true:*

(i) *At least $\frac{r}{d}$ elements of a geometric r-cycle must be distinct.*

(ii) $G_r := \{x_0, x_1, \cdots, x_{r-1}\}$ *is a geometric* r*-cycle of the* p*-periodic difference equation in (1.5), if and only if,* G_r *is a geometric* r*-cycle of each of the* $\frac{p}{d}$ d*-periodic equations*

$$x_{n+1} = f_{(n \bmod d)+dj}(x_n), \quad j = 0, 1, \ldots, \frac{p}{d} - 1.$$

In particular, if r *and* p *are relatively prime; i.e.* $d = 1$*, then* G_r *is an* r*-cycle of each one of the maps* $f_i, 0 \leq i \leq p-1$*.*

(iii) *If* $\{x_0, x_1, \cdots, x_{r-1}\}$ *is an* r*-cycle of each one of the maps* $f_j, 0 \leq j \leq p-1$*, then there exist at least* $\gcd(r, p)$ *geometric* r*-cycles, namely,*

$$\{x_{jd+i \bmod r}, x_{jd+i+1 \bmod r}, \cdots x_{jd+i-1 \bmod r}\}, \ 0 \leq j \leq \frac{r}{d} - 1, 0 \leq i \leq d-1.$$

3 Periodic iterations of polynomials

We motivate our discussion in this section by the simple logistic equation $x_{n+1} = \mu x_n(1 - x_n)$, see May [30,31]. For $\mu > 1 + \sqrt{8}$, the logistic equation has a periodic orbit of minimal period 3; consequently, each positive integer is the minimal period of a periodic orbit. Under periodic forcing, the logistic model takes the form

$$x_{n+1} = \mu_{n \bmod p} x_n(1 - x_n) \tag{3.1}$$

and it is known [2] that the cascade of periods is given by

$$3p \prec 5p \prec \cdots \prec 2{\cdot}3p \prec 2{\cdot}5p \prec \cdots \prec \cdots 2^n{\cdot}3p \prec 2^n{\cdot}5p \prec \cdots \prec 2^2p \prec 2p \prec p.$$

The periodic logistic model in Eq. (3.1) does not exhibit r-cycles for any $r \neq mp, \ m \in \mathbb{N}$. However, if we consider another quadratic form of the logistic equation, then r-cycles, $r \neq mp, \ m \in \mathbb{N}$ appear as we clarify further in the sequel. This discussion leads us to investigate the Γ-set for polynomials; we begin with an important lemma relating the periods of geometric cycles with the degrees of the polynomials in the p-periodic sequence in (1.4).

Lemma 3.1 *Let the elements of the* p*-periodic sequence in (1.4) be nonconstant polynomials. If there exists a geometric* r*-cycle that is not a multiple of* p*, then*

$$\frac{r}{\gcd(r, p)} \leq \max_{0 \leq i \leq p-1} deg(f_i),$$

where $deg(f_i)$ denotes the degree of the polynomial f_i.

PROOF. Assume r is not a multiple of p and $r/\gcd(r,p) > \max_{0\leq i\leq p-1} deg(f_i) =: M$. Obviously, $\mathrm{lcm}(r,p)/p > M$. Now, define $d := \gcd(r,p)$, for each $0 \leq j \leq d-1$, the polynomials

$$f_j, f_{d+j}, f_{2d+j}, \cdots, f_{(\frac{p}{d}-1)d+j}$$

intersect on at least $M+1$ points (see the orbit in (2.1)). Thus $f_j = f_{d+j} = f_{2d+j} = \cdots = f_{(\frac{p}{d}-1)d+j}$ for each $0 \leq j \leq d-1$, and that contradicts the minimality of p.

Remark 3.2 *The result of lemma 3.1 can be improved if there are common factors between the polynomials f_i. For instance, the 6-periodic logistic equation $x_{n+1} = \mu_n x_n(1-x_n)$ cannot have a geometric 4-cycle, even though $\frac{4}{gcd(4,6)} \leq \max_i \; deg(f_i)$. However, the 6-periodic equation*

$$x_{n+1} = \alpha_n x_n^2 + \beta_n x_n + \gamma_n, \; \alpha_{n+6} = \alpha_n, \; \beta_{n+6} = \beta_n, \; \gamma_{n+6} = \gamma_n, \; \forall n \in \mathbb{N}$$

can have a geometric 4-cycle. For instance, consider

$$(\alpha_i, \beta_i, \gamma_i) = \begin{cases} (1+i, -i, 0), \; i = 0, 2, 4 \\ (-i, i-1, 1), \; i = 1, 3, 5. \end{cases}$$

$G = \{0, 0, 1, 1\}$ *is a geometric 4-cycle.*

The next two corollaries are consequences of this lemma.

Corollary 3.3 *Let the elements of the p-periodic sequence in (1.4) be polynomials such that $\max_{0\leq i\leq p-1} deg(f_i) = M$. The Γ-set is finite.*

Corollary 3.4 *Let the elements of the p-periodic sequence in (1.4) be polynomials such that $\max_{0\leq i\leq p-1} deg(f_i) = M$. If there exists a geometric r-cycle, then one of the following holds true:*

(i) *r is a multiple of p.*
(ii) *$r \leq \frac{Mp}{2}$ if p is even and $r \leq \frac{M(p-1)}{2}$ if p is odd.*

PROOF. Suppose r is not a multiple of p; then

$$1 \leq \gcd(p, r) \leq \begin{cases} \frac{p}{2} & \text{if } p \text{ is even} \\ \frac{p-1}{2} & \text{if } p \text{ is odd.} \end{cases} \tag{3.2}$$

From this inequality and the inequality in Lemma (3.1), the result is obtained.

We clarify the significance of the results discussed in this section by giving concrete examples. Consider the functions $f_j,\ 0 \leq j \leq p-1$ to be quadratic, and cubic polynomials. We list the possibilities of r and Γ for different values of p in Table 1 and Table 2. Here, we must stress that we are neglecting the possibility of having more than one geometric cycle of the same period.

Table 1
The possible structure of Γ when the elements of the p-periodic sequence in (1.4) are quadratic polynomials

$f_0, f_1, \cdots, f_{p-1}$ are quadratic polynomials		
p	r	Γ
2	1	$\{1\}$
3	1,2	$\{1\}, \{2\}$
4	1,2	$\{1\}, \{2\}, \{1, 2\}$
5	1,2	$\{1\}, \{2\}$
6	1,2,3,4	$\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}$
7	1,2	$\{1\}, \{2\}$
8	1,2,4	$\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}$
9	1,2,3,6	$\{1\}, \{2\}, \{3\}, \{6\}, \{1, 3\}$
10	1,2,4,5	$\{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 5\}, \{4, 5\}$

4 Periodic iterations of rational functions

Before we generalize the results of the previous section, let us again motivate the discussion by contemplating a prototype of first order rational difference equations. Consider the well-known Beverton-Holt model [4]

$$x_{n+1} = \frac{\mu K}{K + (\mu - 1)x_n} x_n,\ \mu > 1,\ K > 0,\ n \in \mathbb{N},$$

where K is the carrying capacity and μ is the inherent growth rate. In a periodically fluctuating habitat, the Beverton-Holt model takes the form

Table 2
The possible structure of Γ when the elements of the p-periodic sequence in (1.4) are cubic polynomials

$f_0, f_1, \cdots, f_{p-1}$ are cubic polynomials		
p	r	Γ
2	1,3	$\{1\}, \{3\}$
3	1,2	$\{1\}, \{2\}, \{1,2\}$
4	1,2,3,6	$\{1\}, \{2\}, \{3\}, \{6\}, \{1,2\}$
5	1,2,3	$\{1\}, \{2\}, \{3\}, \{1,2\}$
6	1,2,3,4,9	$\{1\}, \{2\}, \{3\}, \{4\}, \{9\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{2,9\}, \{3,4\}, \{4,9\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}$
7	1,2,3	$\{1\}, \{2\}, \{3\}, \{1,2\}$
8	1,2,3,4,6,12	$\{1\}, \{2\}, \{3\}, \{4\}, \{6\}, \{12\}$ $\{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}$
9	1,2,3,6	$\{1\}, \{2\}, \{3\}, \{6\}, \{1,2\}, \{1,3\}, \{1,6\}, \{2,3\}, \{3,6\}, \{1,2,3\}, \{1,2,6\}, \{2,3,6\}$
10	1,2,3,4,5,6,15	$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{15\}, \{1,2\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{2,15\}, \{4,5\}, \{5,6\}, \{6,15\}, \{1,2,5\}, \{1,2,15\}, \{2,4,5\}, \{2,4,15\}$

$$x_{n+1} = \frac{\mu K_{n \bmod p}}{K_{n \bmod p} + (\mu - 1)x_n} x_n, \ n \in \mathbb{N}. \tag{4.1}$$

It is well-known that Eq. (4.1) has a globally asymptotic stable cycle [9–12], and thus, it is the unique cycle. We can write Eq. (4.1) as

$$x_{n+1} = \frac{\mu}{1 + \frac{(\mu-1)}{K_{n \bmod p}} x_n} x_n, \ n \in \mathbb{N}. \tag{4.2}$$

Here, the functions f_j of Eq. (1.4) are rational functions with common numerators μx and denominators $1 + \frac{(\mu-1)}{K_j} x$. Since the numerators are common between all f_j, the denominators are the deciding factor in forming the Γ-set. Indeed, since $y_j = 1 + \frac{(\mu-1)}{K_j} x$ do not intersect in the positive quadrant, the Γ-set is empty. However, although μ is a characteristic of the population, if we allow μ to fluctuate periodically, then the lines $y_j = 1 + \frac{(\mu_j-1)}{K_j} x$ can intersect in the positive quadrant, which may force the Γ-set to be inhabited by at most one element.

Now, we are ready to reinforce our discussion with a solid theory. Consider the maps in (1.4) to be rational functions, i.e., for $i = 0, \dots, p-1$,

$$f_i(x) = \frac{\sum_{j=0}^{k_i} a_{j,i} x^j}{\sum_{j=0}^{m_i} b_{j,i} x^j}.$$

For each i such that $0 \le i \le p-1$, define m_i, M_i to be the degrees of the numerator and denominator of f_i respectively. The next theorem generalizes Lemma 3.1.

Theorem 4.1 *Let the elements of the p-periodic sequence in (1.4) be rational functions and define $M := \max\{m_i + M_j : \ 0 \le i, j \le p-1\}$. If there exists a geometric r-cycle, then $\frac{r}{\gcd(r,p)} \le M$.*

PROOF. Observe that if two of the rational functions $f_i, \ i = 0, \dots p-1$ intersect in $M+1$ points, then they must be identical. The rest of the proof repeats the same argument as in the proof of Lemma 3.1 and Corollary 3.4; thus, we omit it.

As in the previous section, the next corollary is straightforward.

Corollary 4.2 *Let the elements of the p-periodic sequence in (1.4) be rational functions and define $M := \max\{m_i + M_j : \ 0 \le i, j \le p-1\}$. If there exists a geometric r-cycle, then either r is a multiple of p, or $r \le M\frac{p}{2}$ if p is even and $r \le M\frac{p-1}{2}$ if p is odd.*

Remark 4.3 *As in Remark 3.2, if the numerators (or denominators) of the functions f_i share common factors, then Theorem 4.1 can be improved. If there is a common factor of order k, then the value of M in Theorem 4.1 becomes* $\max\{m_i + M_j - k : \ 0 \le i, j \le p-1\}$.

We show the elegance of Theorem 4.1 and Remark 4.3 in the next section.

5 Applications

As we have already discussed the logistic and Beverton-Holt equations in a periodically fluctuating environment, in this section, we apply the techniques of the previous sections to several known equations and models from the literature [8,13,20–22,29,34].

5.1 Ricatti Equation

The autonomous Riccati equation is given by $x_{n+1}=\frac{\alpha+\beta x_n}{A+Bx_n}$ [29]. Now, consider the p-periodic Riccati equation

$$x_{n+1}=\frac{\alpha_n+\beta_n x_n}{a_n+b_n x_n},\quad \alpha_n,\beta_n,a_n,b_n\in\mathbb{R}^+. \tag{5.1}$$

It is straightforward to say that the p-periodic Ricatti equation cannot have r-cycles if $r>2gcd(r,p)$. However, by using the substitution

$$x_n=\frac{\beta_n+a_n}{b_n}y_n-\frac{a_n}{b_n},$$

the above equation reduces to

$$y_{n+1}=\frac{\delta_n y_n-\mu_n}{y_n}$$

where

$$\mu_n=\frac{b_{n+1}(\beta_n a_n+\alpha_n b_n)}{b_n(\beta_{n+1}+a_{n+1})(\beta_n+a_n)}\quad\text{and}\quad \delta_n=\frac{b_{n+1}\beta_n+b_n a_{n+1}}{b_n(\beta_{n+1}+a_{n+1})}.$$

Since the denominator is in common between all the functions f_j, then we cannot have r-cycles for $r\neq gcd(r,p)$. On the other hand, the autonomous Riccati equation with positive coefficients has fixed points only. Thus, the p-periodic Riccati equation has possible r-cycles if r divides p only.

5.2 Hassell's Model

Consider the single-species population model given by

$$x_{n+1}=\frac{\lambda x_n}{(1+ax_n)^b},\ n\in\mathbb{N} \tag{5.2}$$

where λ is the finite rate of increase and a,b are constants defining the density dependent feedback term [21,22]. Forcing λ to be periodic does not help the Γ-set to become inhabited, so we force λ and a to be periodic of common period $p>1$ and keep $b\in\mathbb{Z}^+$ constant. Because x in the numerator is in common between all $f_j,\ 0\leq j\leq p-1$, we neglect it. By Theorem 4.1, any $r\in\Gamma$ must satisfy $r\leq b\,\text{gcd}(r,p)$; however, since $\left(\lambda_j^{\frac{1}{b}}/(1+a_jx)\right)^b$, then a fixed b has no role in changing the structure of Γ. Thus $r\leq\text{gcd}(r,p)$; consequently, the Γ-set can be inhabited by at most one element which is a divisor of p. Finally, let b fluctuate periodically

$$x_{n+1} = \frac{\lambda_n x_n}{(1 + a_n x_n)^{b_n}},\ \lambda_{n+p} = \lambda_n,\ a_{n+p} = a_n,\ b_{n+p} = b_n\ n \in \mathbb{N} \tag{5.3}$$

and define $m = \max\{b_0, b_1, \ldots, b_{p-1}\}$. By Corollary 4.2, Γ can be inhabited with elements r such that $r \leq bp$ or $b(p-1)$. This result can be improved by understanding the nature of the parameters λ, a and b.

5.3 Smith-Slatkin Model

Maynard Smith and Slatkin [33] (See [32] also) considered the difference equation

$$x_{n+1} = \frac{K^c \mu x_n}{K^c + (\mu - 1)x_n^c},\ K, c > 0,\ \mu > 1,\ n \in \mathbb{N} \tag{5.4}$$

to model a prey species (in the absence of predators) that is born in one summer, survives the winter to breed in the next summer and then dies. Observe that if $c = 1$, then the Smith-Slatkin model reduces to the classical Beverton-Holt model, which we discussed in Section 4. Therefore, let us force periodicity on μ and K and keep $c > 1$ a fixed positive integer

$$x_{n+1} = \frac{K_n^c \mu_n x_n}{K_n^c + (\mu_n - 1)x_n^c},\ K_{n+p} = K_n,\ \mu_{n+p} = \mu_n, \forall n \in \mathbb{N}. \tag{5.5}$$

$z := K_n^c + (\mu_n - 1)x^c$ is the non-common factor between the denominators of the maps f_j in Eq. (1.5). Let $y = x^c$, $z = K_n^c + (\mu_n - 1)y$, such lines can intersect at no more than one point. Thus, under these circumstances, the Γ-set can be inhabited by at most one element which is a divisor of the phase period p regardless of the value of c. Now, allow c to fluctuate periodically

$$x_{n+1} = \frac{K_n^{c_n} \mu_n x_n}{K_n^{c_n} + (\mu_n - 1)x_n^{c_n}},\ K_{n+p} = K_n,\ \mu_{n+p} = \mu_n,\ c_{n+p} = c_n,\ \forall n \in \mathbb{N}. \tag{5.6}$$

Define $c := \max\{c_0, c_1, \ldots, c_{p-1}\}$. From Corollary 4.2 and Remark 4.3, the Γ-set can be inhabited with elements r such that $r \leq cp$ or $c(p-1)$.

5.4 Bobwhite Quail Model

Milton and Belair [34] used a hump-with-tail model

$$x_{n+1} = \alpha x_n + \frac{\beta x_n}{1+x_n^\gamma}, \quad \alpha+\beta>1>\alpha>0,\ \gamma>0 \tag{5.7}$$

to describe the growth of the bobwhite quail population in Wisconsin. This model has the capability of producing complex behavior, especially when the steepness of the map is increased by allowing the parameter γ to be sufficiently large. Without being concerned with which parameters are characteristics of the environment and which are demographic characteristics of the species, we allow the parameters to fluctuate periodically. If β is periodic of minimal period p, i.e., $\beta_{n+p} = \beta_n$, then we obtain

$$x_{n+1} = \frac{(\alpha(1+x_n^\gamma)+\beta_n)x_n}{1+x_n^\gamma}, \quad \alpha+\beta_n>1>\alpha>0,\ \gamma>0 \tag{5.8}$$

Since $z := \alpha(1+x^\gamma)+\beta_n$ is the noncommon factor, by Theorem 4.1 and Remark 4.3, $M=\gamma$. However, the role of β_n in this factor is limited to vertical translations, and thus, it does not help the Γ-set to become inhabited. Now, let us allow α to be periodic so that β and α have a common period $p>1$,

$$x_{n+1} = \frac{(\alpha_n(1+x_n^\gamma)+\beta_n)x_n}{1+x_n^\gamma}, \quad \alpha_n+\beta_n>1>\alpha_n>0,\ \gamma>0. \tag{5.9}$$

Again here, $z := \alpha_n(1+x^\gamma)+\beta_n$ is the only factor not in common, let $1+x^\gamma = y$, then α and β play the same role as in $y = \alpha y+\beta$. Thus, the periodicity of α and β forces the Γ-set to be inhibited by at most one element which is a divisor of the phase period p. Finally, let us allow γ to fluctuate periodically so that α, β and γ have a common period $p>1$,

$$x_{n+1} = \frac{(\alpha_n(1+x_n^{\gamma_n})+\beta_n)x_n}{1+x_n^{\gamma_n}}, \quad \alpha_n+\beta_n>1>\alpha_n>0,\ \gamma_n>0. \tag{5.10}$$

Define $\gamma := \max\{\gamma_0, \gamma_1, ...\gamma_{p-1}\}$, from Theorem 4.1 and Remark 4.3, the Γ-set can be inhibited with elements r such that $r \le \gamma p$ or $\gamma(p-1)$. Also here, the result can be improved by understanding the nature of the parameters α, β, and γ.

We close this section with the following remark.

Remark 5.1 *It is worth mentioning that it is possible to formulate the theory of this paper in terms of the free parameters in the rational functions* f_j, $0 \le$

$j \leq p-1$. *However, we find it more convenient to develop the theory in terms of degrees of polynomials.*

6 Structural stability

Let I be a compact interval and $\mathcal{P}(I)$ be the space of polynomials with real coefficients. Weierstrass's theorem states that if $f \in \mathcal{C}(I)$, then f can be uniformly approximated by polynomials in $\mathcal{P}(I)$. The discussion provided above and Weierstrass's theorem motivate further investigation of the structure of Γ when $f_0, f_1, \ldots, f_{p-1}$ are elements of $\mathcal{C}(I)$. A simple comparison between the autonomous logistic equation $x_{n+1} = \mu x_n(1-x_n)$ and the nonautonomous p-periodic equation $x_{n+1} = \mu_n x_n(1-x_n),\ \ \mu_n = \mu + \epsilon_n$ suggests that the cycle structure of Eq. (1.3) is unstable. In other words, if we approximate each continuous map $f_j,\ 0 \leq j \leq p-1$ by a polynomial P_j, then the Γ-set of $x_{n+1} = P_n(x_n)$ is finite, while the Γ-set of $x_{n+1} = f_n(x_n)$ could be infinite. This leads us to construct an example of a p-periodic difference equation in which $\{f_0, f_1, \ldots, f_{p-1}\} \subseteq \mathcal{C}(I)$ and the cardinality of Γ is infinite.

Example 6.1 *Let $I = [0,1]$ and define the map*

$$f_0(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} < x \leq \frac{1}{4} \\ \frac{18}{5}x - \frac{7}{10} & \frac{1}{4} < x \leq \frac{1}{3} \\ x + \frac{1}{6} & \frac{1}{3} < x \leq \frac{1}{2} \\ -2x + \frac{5}{3} & \frac{1}{2} < x \leq \frac{2}{3} \\ -x+1 & \frac{2}{3} < x \leq 1. \end{cases}$$

Observe that $\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$ is a 3-cycle for the autonomous difference equation $x_{n+1} = f_0(x_n)$. By Sharkovsky's theorem, it has r-cycles for all $r \in \mathbb{Z}^+$. Furthermore, each point of the interval $(\frac{1}{5}, \frac{1}{4}]$ is eventually fixed point, and thus does not belong to any periodic orbit. Now, we define the maps $f_1, \ldots, f_{p-1}$ as

$$f_j(x) = \begin{cases} f_0(x) & x \in [0, \frac{1}{5}] \cup [\frac{1}{4}, 1] \\ \left(\frac{1600}{4^j} - 480\right)x^2 + \left(\frac{-720}{4^j} + 216\right)x + \frac{80}{4^j} - \frac{119}{5} & \frac{1}{5} < x < \frac{1}{4}. \end{cases}$$

From Proposition 2.2, the p-periodic equation in (1.3) *has at least* $\gcd(r,p)$ *r-cycles for each $r \in \mathbb{Z}^+$. Thus, Γ has infinite cardinality.*

Although Example 6.1 clarifies the notion of structural stability as posed in this section, it motivates another question. Can the Γ-set be large in the sense of cardinality, while the maps $f_j,\ 0 \le j \le p-1$ agree on a set small in the sense of Lebesgue measure. This would be the topic of further abstract research on this subject.

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