

# An Extension of Sharkovsy's Theorem to Periodic Difference Equations

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## Abstract

We present an extension of Sharkovsky's Theorem and its converse to periodic difference equations. In addition, we provide a simple method for constructing a  $p$ -periodic difference equation having an  $r$ -periodic geometric cycle with or without stability properties.

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## 1 Introduction.

Consider the nonautonomous  $p$ - difference equation

$$x_{n+1} = F(n, x_n) \tag{1}$$

where  $F : \mathbb{Z}^+ \times X \rightarrow X$  is a continuous map,  $X$  is a metric space such that  $F(n + p, \cdot) = F(n, \cdot)$  for all  $n \in \mathbb{Z}^+$ . The period  $p \geq 2$  is assumed to be minimal. Equation (1) may be written in the more convenient form

$$x_{n+1} = f_n(x_n) \tag{1'}$$

where the map  $f_n = F(n, \cdot)$  for each  $n \in \mathbb{Z}^+$ . Thus  $f_{n+p} = f_n$  for all  $n \in \mathbb{Z}^+$ .

The study of periodic difference equations was initiated in the mathematical biology literature almost a quarter century ago by Coleman [3], Jillson [22], and Kapral and Mandel [23]. The subject stays moribund until it picked up

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steam in the late 90's and early 2000 with the appearance of the papers by Cushing and Henson [4–6], Henson [21], Clark and Gross [2], Grinfeld, Knight and Lamba [20], Frank and Selgrade [18], Frank and Yakubu [19], Selgrade and Roberdo [32], Jia Li [29], Kon [26,27], Kocic [24], Elaydi and Sacker [12–16].

The main objective here is to extend the well-known Sharkovsky's Theorem and its converse (Sharkovsky [34], Li and Yorke [30], Elaydi [9–11]) to periodic difference equations (Section 3). To accomplish this task we introduce a new ordering of the positive integers which will be given the name of “ $p$ -Sharkovsky's ordering.” But before embarking on this task we need to discuss the notion of periodic orbits or cycles for Equation (1). In [12], the notion of geometric cycles was introduced as a natural extension of the simple notion of periodic cycles in autonomous systems. It is the authors' point of view that the notion of geometric cycles is better understood via the construction of an associated skew-product dynamical systems. As the notion of geometric cycles have not taken root, we revisit it again and define a new equivalence relation among the various cycles. Given two positive integers  $r$  and  $p$  with  $r|p$  ( $r$  divides  $p$ ), Elaydi and Sacker [12] constructed a  $p$ -periodic difference equation that has a globally asymptotically stable  $r$ -periodic cycle. In Section 2 we present a much simpler construction for  $p$ -periodic difference equations with a globally asymptotically stable, locally asymptotically stable, and unstable  $r$ -periodic cycle. In addition, we indicate how to construct  $p$ -difference equations with multiple attracting periodic cycles.

## 2 Geometric cycles and skew-product dynamical systems

Let  $Y = \{f_0, f_1, \dots, f_{p-1}\}$ . Consider the map  $\pi : X \times Y \times \mathbb{Z}^+ \rightarrow X \times Y$  defined by  $\pi((x_0, f_i), n) = (\Phi_n(f_i)x_0, f_{i+n})$ , with

$$\Phi_n(f_i) = f_{i+n-1} \circ f_{i+1} \circ f_i. \quad (2)$$

Then  $\pi$  is a skew-product semidynamical system [12,16] on the product space as depicted in the following diagram.

$$\begin{array}{ccc} X \times Y \times \mathbb{Z}^+ & \xrightarrow{\pi} & X \times Y \\ \downarrow p \times id & & \downarrow p \\ Y \times \mathbb{Z}^+ & \xrightarrow{\sigma} & Y \end{array}$$

Fig. 1.  $\sigma$  is the shift map  $\sigma(f_i, n) = f_{i+n}$ , and  $p$  is the projection map

For each  $f_i$ ,  $0 \leq i \leq d-1$ , the set  $p^{-1}(f_i) = \mathcal{F}_i$  is called the  $i$ th fiber in the product space  $X \times Y$ , where  $p(x, y) = y$  is the projection map. .

We are now ready to introduce the notion of “geometric cycles”

**Definition 1** Let  $c_r = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$  be a set of points in the state space  $X$ , with  $r \geq 1$ . Then  $c_r$  is said to be a geometric  $r$ -cycle if for  $i = 0, 1, \dots, r-1$ ,

$$f_{(i+nr) \bmod p}(\bar{x}_i) = \bar{x}_{(i+1) \bmod r}, \quad n \in \mathbb{Z}^+. \quad (3)$$

**Standing Notation.** Let  $s = \text{lcm}(p, r)$  denote the least common multiple of  $p$  and  $r$ ,  $d = \text{gcd}(p, r)$  denote the greatest common divisor of  $p$  and  $r$ ,  $\ell = \frac{s}{p}$ , and  $m = \frac{p}{d}$ . Then it is easy to see that  $\ell = \frac{r}{d}$ .

Both integers  $\ell$  and  $d$  play a fundamental role in our understanding of geometric cycles. It was shown in [14] that  $c_r$  is a geometric  $r$ -cycle if and only if

$$f_{(i+nr) \bmod p}(\bar{x}_i) = \bar{x}_{(i+1) \bmod r}, \quad 0 \leq i \leq r-1. \quad (4)$$

Furthermore, the orbit of  $(\bar{x}_0, f_0)$  in the skew-product space  $\pi$  intersects each fiber  $\mathcal{F}_i$  in exactly  $\ell$  points [12,14]. Now suppose that  $c_r = c_{r,0} = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$  is a geometric  $r$ -cycle. Then the fiber  $\mathcal{F}_0$  “contains” the  $\ell$  points  $\bar{x}_0, \bar{x}_d, \bar{x}_{2d \bmod r}, \dots, \bar{x}_{(\ell-1)d \bmod r}$ .

### 3 Construction of geometric cycles

Given two positive integers  $r \geq 1$  and  $p > 1$  with  $r|p$  ( $r$  divides  $p$ ), Elaydi and Sacker [12] constructed a  $p$ -periodic difference equation that has a globally asymptotically stable geometric  $r$ -cycle. The condition that  $r|p$  is based on Elaydi-Sacker Theorem which we now state for convenience of the reader.

**Theorem 2** Consider the difference equation (1) with a minimal period  $p$  such that each  $f_i : X \rightarrow X$  is a continuous function on a connected metric space. If  $c_r$  is a geometric  $r$ -cycle which is globally asymptotically stable, then  $r|p$ .

In this section we will present a more general construction of  $p$ -periodic difference equations with geometric  $r$ -cycles with or without stability properties. Moreover, our construction procedure is simpler than that reported in [12].

**Theorem 3** For any given positive integers  $p > 1$  and  $r \geq 1$ , there exist polynomials  $f_0, f_1, \dots, f_{p-1}$  of degree at most  $\ell := \frac{\text{lcm}(p,r)}{p}$  such that the  $p$ -periodic difference equation  $x_{n+1} = f_n(x_n)$  has a geometric  $r$ -cycle.

**PROOF.** Assume the geometric  $r$ -cycle to be constructed is

$$\bar{x}_0 \rightarrow \bar{x}_1 \rightarrow \bar{x}_2 \rightarrow \dots \rightarrow \bar{x}_{r-1},$$

with distinct  $\bar{x}_i$ 's if necessary. For each  $0 \leq i \leq d-1$  find the unique interpolating polynomial  $f_i$  of degree at most  $\ell-1$  passing through the points  $(\bar{x}_i, \bar{x}_{i+1}), (\bar{x}_{i+d}, \bar{x}_{i+d+1}), \dots, (\bar{x}_{i+(\ell-1)d \bmod r}, \bar{x}_{i+1+(\ell-1)d \bmod r})$  (see [25] for background information on interpolation).  $f_i(x)$  can be written as

$$f_i(x) = \sum_{j=0}^{\ell-1} \bar{x}_{i+1+jd \bmod r} \ell_j(x), \quad (5)$$

where  $\ell_j(x)$  satisfies  $\ell_j(\bar{x}_{i+kd \bmod r}) = 0$ ,  $j \neq k$  and  $\ell_j(\bar{x}_{i+jd \bmod r}) = 1$ .  $\ell_j(x)$  are the Lagrange bases polynomials associated with the nodes  $\{\bar{x}_i, \bar{x}_{i+d}, \dots, \bar{x}_{i+(\ell-1)d \bmod r}\}$ , which can be written as

$$\ell_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^{\ell-1} \frac{x - \bar{x}_{(i+kd) \bmod r}}{\bar{x}_{(i+jd) \bmod r} - \bar{x}_{(i+kd) \bmod r}}. \quad (6)$$

Clearly,

$$f_i(\bar{x}_{(i+kd) \bmod r}) = \bar{x}_{(i+1+kd) \bmod r}, \quad 0 \leq k \leq \ell-1.$$

Next, fix  $0 \leq i \leq d-1$  and for each  $1 \leq j \leq m-1$  build the mappings  $f_{i+jd}(x)$  as

$$f_{i+jd}(x) = f_i(x) + \lambda_j \prod_{k=0}^{\ell-1} (x - \bar{x}_{i+kd \bmod r}), \quad (7)$$

where the parameters  $\lambda_j$ ,  $1 \leq j \leq m-1$  can be manipulated freely to keep the system in nonautonomous form. Observe that the polynomials  $f_i(x)$ ,  $0 \leq i \leq d-1$  are of degree at most  $\ell-1$ , and for all  $0 \leq i \leq d-1$  and  $1 \leq j \leq m-1$ ,  $f_{i+jd}(x)$  are of degree at most  $\ell$ . Also, it is a straight forward substitution to find that for all  $n \in \mathbb{N}$ ,

$$f_{n \bmod p}(\bar{x}_{n \bmod r}) = \bar{x}_{n+1 \bmod r}.$$

**Remark 4** (i) *Polynomial interpolation was used in Theorem 3. In fact, piecewise linear interpolation or Splines can be used as well.*

(ii) *Since polynomials of lower degree are best used in calculations, we confined ourselves with polynomial of degree at most  $\frac{\text{lcm}(p,r)}{p}$  in the proof of Theorem 3. However, by neglecting the degree of the constructed polynomials, the algorithm provided can be made simpler. We define  $f_0(x)$  as*

$$f_0(x) = \sum_{j=0}^{r-1} \bar{x}_{j+1 \bmod r} \prod_{i=0, i \neq j}^{r-1} \frac{x - \bar{x}_i}{\bar{x}_j - \bar{x}_i} \quad (8)$$

and for all  $0 \leq n \leq p - 1$  define

$$f_n(x) := f_0(x) + \lambda_n \prod_{i=0}^{r-1} (x - \bar{x}_i). \quad (9)$$

**Corollary 5** For any positive integers  $r \geq 1$  and  $p > 1$ , there exists a  $p$ -periodic difference equation with the following properties

- (i) there exists a geometric  $r$ -cycle,
- (ii) there are no  $k$ -cycles for any  $k \notin \mathcal{B} := \{r, mp : m \in \mathbb{Z}^+\}$ ,

**PROOF.** To simplify the work, we follow the above remark, and without loss of generality, assume the  $r$ -cycle to be constructed is

$$0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow r - 2 \rightarrow r - 1.$$

Then

$$\begin{aligned} f_0(x) &= \sum_{j=0}^{r-1} (j+1) \prod_{i=0, i \neq j}^{r-1} \frac{(x-i)}{(j-i)}, \\ f_n(x) &= f_0(x) + \frac{1}{n} \prod_{i=0}^{r-1} (x-i), \quad 1 \leq n \leq p-1, \\ f_n(x) &= f_{n \bmod p}(x), \quad n \geq p. \end{aligned}$$

Now, for  $r > 1$  it is obvious that no equilibrium points exist. To show the non existence of  $k$ -cycles for  $k \notin \mathcal{B}$ , suppose there exists a nontrivial  $k$ -cycle,  $k \notin \mathcal{B}$ , say

$$\bar{x}_0 \rightarrow \bar{x}_1 \rightarrow \bar{x}_2 \rightarrow \cdots \rightarrow \bar{x}_{k-2} \rightarrow \bar{x}_{k-1}.$$

As before, Let  $s := \text{lcm}(p, k)$ . Then we require the following  $s$  equations

$$\bar{x}_{n+1 \bmod k} = f_n(\bar{x}_{n \bmod k}), \quad 0 \leq n \leq s - 1$$

to be satisfied.

But at  $n = 0$  and  $n = k$ ,  $\bar{x}_1 = f_0(\bar{x}_0)$  and

$$\bar{x}_1 = f_{k \bmod p}(\bar{x}_0) = f_0(\bar{x}_0) + \frac{1}{k \bmod p} \prod_{i=0}^{r-1} (\bar{x}_0 - i),$$

which implies  $\bar{x}_0 = i$  for some  $i \in \{0, 1, \dots, r-1\}$ . Repeating the same argument at  $n = j$  and  $n = k + j$ ,  $1 \leq j \leq k - 1$  we get a contradiction.

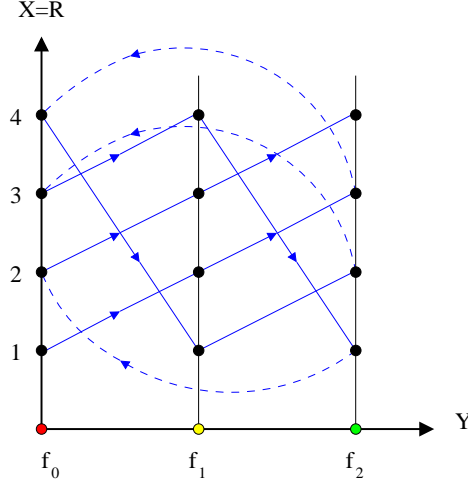


Fig. 2. A 4-cycle in a 3-periodic difference equation. Since  $\ell = 4$ , there are 4 points from the cycle on each fiber.

**Example 6** We construct a 3-periodic difference equation, which has the 4-cycle  $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ . Define

$$f_0(x) = 2 + (x - 1) - \frac{2}{3}(x - 1)(x - 2)(x - 3)$$

$$f_1(x) = 2 + (x - 1) - \frac{2}{3}(x - 1)(x - 2)(x - 3) + (x - 1)(x - 2)(x - 3)(x - 4)$$

$$f_2(x) = 2 + (x - 1) - \frac{2}{3}(x - 1)(x - 2)(x - 3) - (x - 1)(x - 2)(x - 3)(x - 4).$$

Then define

$$f_n := f_{n \bmod 3}, \quad n \geq 3.$$

Observe that  $f_0(1) = 2$ ,  $f_1(2) = 3$ ,  $f_2(3) = 4$ ,  $f_3(4) = f_0(4) = 1$ , and by Corollary 5 there are no  $k$ -cycles for all  $k \in \mathbb{Z} \setminus \{4, 3, 6, 9, 12, \dots\}$  (Fig. 2). Observe that the phase shifts of the cycle  $c_4 = \{1, 2, 3, 4\}$  are given by  $\{3, 4, 1, 2\}$ ,  $\{1, 2, 3, 4\}$ .

To this end we have constructed a  $p$ -periodic difference equation with a geometric  $r$ -cycle for any given positive integers  $r \geq 1$  and  $p > 1$ . Now we will shift our attention on constructing asymptotically stable geometric  $r$ -cycles.

Although one may find the definitions of various notions of stability of nonautonomous difference systems in many books (see [8,28]), nevertheless we found it pedagogically valuable to simplify these notions for periodic systems.

**Definition 7** Let  $c_r = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$  be a geometric  $r$ -cycle in  $X$ . Then

(i)  $c_r$  is uniformly stable if given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any

$n_0 = 0, 1, \dots, p-1$ , and  $x \in X$ ,

$$|x - \bar{x}_{n_0 \bmod r}| < \delta \quad \text{implies} \quad |\Phi_n(f_{n_0})x - \Phi_n(f_{n_0})\bar{x}_{n_0 \bmod r}| < \varepsilon$$

for all  $n \in \mathbb{Z}^+$ , where  $\Phi_n(f_{n_0}) = f_{(n_0+n-1) \bmod p} \circ \dots \circ f_{(n_0+1) \bmod p} \circ f_{n_0}$ .

(ii)  $c_r$  is uniformly attracting if there exists  $\eta > 0$  such that for any  $n_0 = 0, 1, \dots, p-1$ , and  $x \in X$ ,

$$|x - \bar{x}_{n_0 \bmod r}| < \eta \quad \text{implies} \quad \lim_{n \rightarrow \infty} \Phi_{ns}(f_{n_0})x = \bar{x}_{n_0 \bmod r},$$

where  $s = \text{lcm}(r, p)$ .

(iii)  $c_r$  is uniformly asymptotically stable if it is both uniformly stable and uniformly attracting.

**Remark 8** In (i) once you pick  $n_0$ , the  $x$  is located on the fiber  $\mathcal{F}_{n_0}$  along with  $\bar{x}_{n_0}$ . In (ii) each point  $\bar{x}_{n_0}$  in  $c_r$  is a fixed point under the map  $g_{n_0} = \Phi_s(f_{n_0}) = f_{(n_0+s-1) \bmod p} \circ \dots \circ f_{(n_0+1) \bmod p} \circ f_{n_0}$ .

Hence  $\bar{x}_{n_0}$  is asymptotically stable [8,9] if  $|g'_{n_0}(\bar{x}_{n_0})| < 1$ . Consequently,  $\bar{x}_{n_0}$  is asymptotically stable if

$$\left| \prod_{i=0}^{s-1} f'_{(n_0+i) \bmod p}(\bar{x}_{(n_0+i) \bmod r}) \right| < 1.$$

Moreover,  $\bar{x}_{n_0}$  is unstable [8,9] if

$$\left| \prod_{i=0}^{s-1} f'_{(n_0+i) \bmod p}(\bar{x}_{(n_0+i) \bmod r}) \right| > 1.$$

If however,

$$\left| \prod_{i=0}^{s-1} f'_{(n_0+i) \bmod p}(\bar{x}_{(n_0+i) \bmod r}) \right| = 1,$$

then we appeal to Theorems 1.15 and 1.16 in [8].

Using construction (8)-(9), we have the following criterion for stability.

**Proposition 9** *The geometric  $r$ -cycle  $c_r$  is asymptotically stable if*

$$\left| f'_0(\bar{x}_0) \prod_{i=1}^{s-1} \left( f'_0(\bar{x}_{i \bmod r}) + \lambda_{i \bmod p} \prod_{\substack{j=0 \\ j \neq i \bmod r}}^{r-1} (\bar{x}_{i \bmod r} - \bar{x}_j) \right) \right| < 1. \quad (10)$$

This inequality is satisfied if we choose

$$\lambda_i = -\frac{f'_0(\bar{x}_{i \bmod r})}{\prod_{\substack{j=0 \\ j \neq i \bmod r}}^{r-1} (\bar{x}_{i \bmod r} - \bar{x}_j)}, \quad \text{for some } i = 1, 2, \dots, p-1$$

or if we put  $f'_0(\bar{x}_0) = 0$ . The latter condition ( $f'_0(\bar{x}_0) = 0$ ) can be obtained if we use the Hermite interpolation to define  $f_0(x)$ :  $f_0(x) = \sum_{i=0}^{r-1} \bar{x}_{(i+1) \bmod r} [1 - 2(x - \bar{x}_i \ell'_i(\bar{x}_i)) \ell_i^2(x)] \ell_i^2(x)$  where the  $\ell_i$ s are the Lagrange bases given in (6) at the nodes  $\{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$ .

To summarize, we have the following result.

**Theorem 10** *For any given positive integers  $p > 1$  and  $r \geq 1$ , there exists a  $p$ -periodic difference equation  $x_{n+1} = f_n(x_n)$ , which has an asymptotically stable  $r$ -cycle. Furthermore, in our construction there are no  $k$ -cycles for any  $k \notin \{r, 0 \bmod p\}$ .*

**Remark 11** *It is worth mentioning that in Theorems 3 and 10 we confined ourselves with the construction of one cycle, but in fact, the methods in both theorems can be generalized to any finite number of cycles.*

**Example 12 (i)** *Suppose  $r$  and  $p > 1$  are positive integers, and let  $d = \gcd(r, p)$ . Consider  $C_r = \{c_0, c_1, \dots, c_{r-1}\}$ , where  $c_i$ s are different whenever necessary, and define the map  $f_0(x)$  as*

$$f_0(x) = \sum_{i=0}^{r-1} c_{i+1 \bmod r} (1 - 2(x - c_i) \ell'_i(c_i)) \ell_i^2(x), \quad \ell_i(x) := \prod_{j=0, j \neq i}^{r-1} \frac{(x - c_j)}{c_i - c_j}.$$

For  $1 \leq i \leq p-1$ , define

$$f_i(x) = f_0(x) + \frac{1}{i} \prod_{j=0}^{r-1} (x - c_j).$$

Then  $x_{n+1} = f_{n \bmod p}(x_n)$ ,  $n \in \mathbb{N}$  has  $d$  geometric cycles of minimal period  $r$ , namely

$$\{c_0, c_1, \dots, c_{r-1}\}, \{c_1, c_2, \dots, c_{r-1}, c_0\}, \dots, \{c_{d-1}, c_d, \dots, c_{r-1}, c_0, \dots, c_{d-2}\},$$

and each of which is an attractor.

**(ii)** *Given positive integers  $r_1, r_2$  and  $p > 1$ , we construct a  $p$ -periodic difference equation with two attractors*

$$\{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r_1-1}\} \text{ and } \{\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{r_2-1}\}.$$



Assume  $\bar{x}_i$ s and  $\bar{y}_i$ s are different whenever necessary, and define

$$c_i := \begin{cases} \bar{x}_i & 0 \leq i \leq r_1 - 1 \\ \bar{y}_{i-r_1} & r_1 \leq i \leq r_1 + r_2 - 1. \end{cases}$$

Construct a polynomial  $f_0(x)$  such that

$$f_0(c_i) := \begin{cases} c_{i+1 \bmod r_1} & 0 \leq i \leq r_1 - 1 \\ c_{i+1} & r_1 \leq i \leq r_1 + r_2 - 2 \\ c_{r_1} & i = r_1 + r_2 - 1, \end{cases}$$

and  $f_0'(c_i) = 0$  for all  $0 \leq i \leq r_1 + r_2 - 1$ . Such a polynomial can be given by

$$f_0(x) = \sum_{j=0}^{r_1-1} c_{j+1 \bmod r_1} L_j(x) + \sum_{j=r_1}^{r_1+r_2-2} c_{j+1} L_j(x) + c_{r_1} L_{r_1+r_2-1}(x),$$

where

$$L_j(x) := (1 - 2(x - c_j) \mathcal{L}'_j(c_j)) \mathcal{L}_j^2(x) \text{ and } \mathcal{L}_j(x) := \prod_{i=0, i \neq j}^{r_1+r_2-1} \frac{x - c_i}{c_j - c_i}.$$

Finally, define

$$f_n(x) := f_0(x) + \frac{1}{n} \prod_{i=0}^{r_1+r_2-1} (x - c_i) \text{ for all } 1 \leq n \leq p-1.$$

**Remark 13 (A simple construction of globally asymptotically stable geometric cycles)**

Given two positive integers  $p > 1$ ,  $r \geq 2$ , let  $c_r = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$  be a set of points in  $X$ . Define the  $p$  functions  $f_0, f_1, \dots, f_{p-1}$  as follows:

$f_i$  is defined as the line passing through the point  $(\bar{x}_{i \bmod r}, \bar{x}_{(i+1) \bmod r})$  and with slope  $m_i = \frac{1}{p+1} \left[ \frac{i}{r} \right]$ , where  $[\cdot]$  is the greatest integer function.

Then every point in  $X$  is eventually periodic relative to  $c_r$ , and thus  $c_r$  is (trivially) globally asymptotically stable in the  $p$ -periodic difference equation  $x_{n+1} = f_n(x_n)$ ,  $f_{n+p} = f_n$ .

#### 4 Sharkovsky's Theorem for periodic difference equations

In this section we will extend the fundamental theorem of Sharkovsky [7,9,31] to  $p$ -periodic difference equations. Let us start with the Sharkovsky's ordering

of the set of positive integers.

$$\begin{aligned}
& 3 \prec 5 \prec 7 \prec \dots \\
& 2 \times 3 \prec 2 \times 5 \prec 2 \times 7 \prec \dots \\
& \vdots \\
& 2^n \times 3 \prec 2^n \times 5 \prec 2^n \times 7 \prec \dots \\
& \vdots \\
& \dots \prec 2^n \prec 2^{n-1} \prec \dots \prec 2 \prec 1.
\end{aligned}$$

**Theorem 14 (Sharkovsky)** *Let  $f : I \rightarrow I$  be a continuous map on a closed interval  $I$ . If  $f$  has a periodic point of period  $k$ , then it has a periodic point of order  $m$  for all  $m$  with  $k \prec m$ .*

Given two positive integers  $p > 1$  and  $q \geq 1$ , then we can write the prime factorization of  $p$  and  $q$  as

$$p = \prod_{i=1}^k p_i^{s_i}, \quad q = \left( \prod_{i=1}^k p_i^{t_i} \right) \left( \prod_{i=1}^m q_i^{t_i^*} \right)$$

where the  $p_i$ s are the distinct prime factors of  $p$  and the  $q_i$ s are the distinct prime factors of  $q$  that are not in common with  $p$ .

Define the set

$$\mathcal{A}_q(p) = \left\{ q \prod_{i=1}^k p_i^{s_i^*} : s_i^* = 0, 1, \dots, s_i \text{ if } t_i = 0, \text{ and } s_i^* = s_i \text{ if } t_i \neq 0. \right\} \quad (11)$$

If we fix  $p$  during our discussion, then we write  $\mathcal{A}_q(p)$  simply as  $\mathcal{A}_q$ .

The next lemma gives a simple description of the sets  $\mathcal{A}_q$ .

**Lemma 15**

$$\mathcal{A}_q = \{n : \text{lcm}(n, p) = pq\}.$$

**PROOF.** From Definition (11) it follows that  $\mathcal{A}_q \subseteq \{n : \text{lcm}(n, p) = pq\}$ . Conversely, let  $m$  be a solution of  $\text{lcm}(m, p) = pq$ . Then  $\frac{mp}{\text{gcd}(m, p)} = pq$  and hence  $m = q \cdot \text{gcd}(m, p)$ . This implies that  $m \in \mathcal{A}_q$  and the lemma is proved.

The algebra of  $\mathcal{A}_q(p)$  for a fixed  $p \geq 1$  is summarized in the following three statements.

**Proposition 16** *For a fixed positive integer  $p > 1$ , we have*

- (i)  $pq \in \mathcal{A}_q$  for all  $q \in \mathbb{Z}^+$ ,
- (ii)  $\mathcal{A}_{q_1} \cap \mathcal{A}_{q_2} = \emptyset$  if and only if  $q_1 \neq q_2$ ,
- (iii)  $\cup_{q \in \mathbb{Z}^+} \mathcal{A}_q = \mathbb{Z}^+$ .

Now to each positive integer  $p \geq 1$ , we associate the following ordering, which we call the  $p$ -Sharkovsky's ordering.

$$\begin{aligned}
& \mathcal{A}_3 \prec \mathcal{A}_5 \prec \mathcal{A}_7 \prec \dots \\
& \mathcal{A}_{2 \cdot 3} \prec \mathcal{A}_{2 \cdot 5} \prec \mathcal{A}_{2 \cdot 7} \prec \dots \\
& \vdots \\
& \mathcal{A}_{2^n \cdot 3} \prec \mathcal{A}_{2^n \cdot 5} \prec \mathcal{A}_{2^n \cdot 7} \prec \dots \\
& \vdots \\
& \dots \prec \mathcal{A}_{2^n} \prec \dots \prec \mathcal{A}_{2^2} \prec \mathcal{A}_2 \prec \mathcal{A}_1.
\end{aligned}$$

In this ordering, we mean by  $\mathcal{A}_{q_1} \prec \mathcal{A}_{q_2}$ ,  $q_1 \prec q_2$  in the original Sharkovsky's ordering and each element of  $\mathcal{A}_{q_1}$  precedes each element of  $\mathcal{A}_{q_2}$  in the  $p$ -Sharkovsky's ordering.

- Example 17** (i) If  $p = 1$  then the 1-Sharkovsky's ordering reduces to the original Sharkovsky's ordering.  
(ii) If  $p = 2^m$  for some positive integer  $m$  then the  $2^m$ -Sharkovsky's ordering simplifies to

$$\begin{aligned}
& 3, 3 \cdot 2, \dots, 3 \cdot 2^m \prec 5, 5 \cdot 2, \dots, 5 \cdot 2^m \prec 7, 7 \cdot 2, \dots, 7 \cdot 2^m \prec \dots \\
& 3 \cdot 2^{m+1} \prec 5 \cdot 2^{m+1} \prec 7 \cdot 2^{m+1} \prec \dots \\
& \vdots \\
& 3 \cdot 2^{m+n} \prec 5 \cdot 2^{m+n} \prec 7 \cdot 2^{m+n} \prec \dots \\
& \vdots \\
& \dots \prec 2^{m+n} \prec \dots \prec 2^{m+2} \prec 2^{m+1} \prec 2^m, 2^{m-1}, \dots, 2^2, 2, 1
\end{aligned} \tag{12}$$

Now we are ready to state Sharkovsky's Theorem for periodic difference equations.

**Theorem 18 (Sharkovsky's Theorem for periodic difference equations)** Suppose that the  $p$ -periodic difference equation  $x_{n+1} = f_n(x_n)$  has a geometric  $r$ -cycle, and let  $\ell := \frac{\text{lcm}(p,r)}{p}$ . Then each set  $\mathcal{A}_q$ , such that  $\mathcal{A}_\ell \prec \mathcal{A}_q$ , contains a period of a geometric cycle.

**PROOF.** Suppose the  $p$ -periodic difference equation  $x_{n+1} = f(n, x_n)$  has a geometric  $r$ -cycle. There are  $\ell$  distinct points in each fiber  $\mathcal{F}_i$ ,  $0 \leq i \leq p-1$ . Furthermore, the  $\ell$  points in  $\mathcal{F}_0$  forms a cycle of period  $\ell$  under the autonomous map  $h(x) = f_{p-1} \circ \dots \circ f_1 \circ f_0(x)$ . Applying Sharkovsky's Theorem for autonomous maps to  $h(x)$ , we conclude that  $h(x)$  has cycles of periods  $q$  for all  $\ell \prec q$  in the  $p$ -Sharkovsky's ordering. Now the  $q$  points of the cycle of period  $q$  are in the intersection of the fiber  $\mathcal{F}_0$  and a geometric  $r^*$ -cycle for some  $r^*$  satisfying the equation  $lcm(p, r^*) = pq$ . By Lemma 15,  $r^* \in \mathcal{A}_q$  and from the definition of the  $p$ -Sharkovsky's ordering each set  $\mathcal{A}_\ell \prec \mathcal{A}_q$  contains a period of a geometric cycle.

**Example 19** We construct the geometric 6-cycle  $\{0, 1, 2, 3, 4, 5\}$  in a 4-periodic difference equation. Since the calculations in high degree polynomials can cause overflow on computers, we depend on Theorem 3 to have polynomials of degree at most 3. Thus we construct  $f_0(x)$  as an interpolating polynomial of the points  $(0, 1)$ ,  $(2, 3)$  and  $(4, 5)$ , and  $f_1(x)$  as an interpolating polynomial of the points  $(1, 2)$ ,  $(3, 4)$ ,  $(5, 0)$ . Simple calculations reveal that  $f_0(x) = x + 1$ , and  $f_1(x) = \frac{-1}{4}(3x - 1)(x - 5)$ . Then construct

$$\begin{aligned} f_2(x) &= f_0(x) + \lambda_2 x(x - 2)(x - 4), & \lambda_2 &\neq 0, \\ f_3(x) &= f_1(x) + \lambda_3(x - 1)(x - 3)(x - 5), & \lambda_3 &\neq 0. \end{aligned}$$

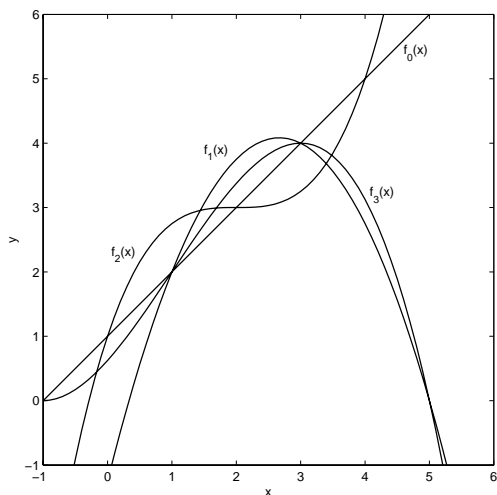


Fig. 3. The graphs of  $f_0(x)$ ,  $f_1(x)$ ,  $f_2(x)$ ,  $f_3(x)$ , where  $\lambda = \frac{1}{4}$  and  $\lambda_3 = \frac{-1}{8}$ .

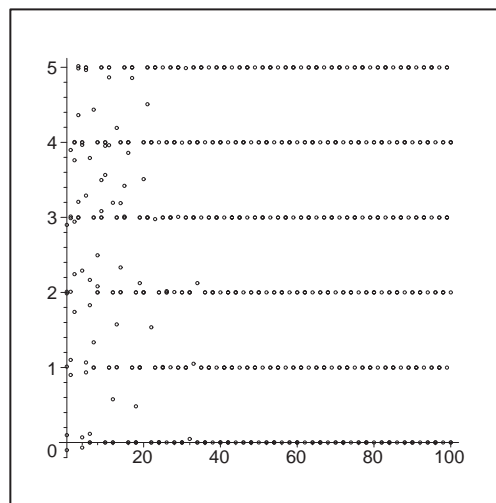


Fig. 4. The plot of the sequence  $f_n(x_n)$ ,  $0 \leq n \leq 100$  for some starting points in the basin of attraction.

Now Eq.  $x_{n+1} = f_{n \bmod 4}(x_n)$  has the following properties.

- (1)  $\{0, 1, 2, 3, 4, 5\}$  is a unique 6-cycle.

- (2) By an argument similar to that in Corollary 5, there are no  $k$ -cycles for all  $k \neq \{6, 4m : m \in \mathbb{Z}^+\}$ .
- (3) By the stability criterion in Proposition 9, we pick  $\lambda_2 = \frac{1}{4}$  and  $\lambda_3 = \frac{-1}{8}$  so the constructed 6-cycle is stable.
- (4) By Sharkovsky's Theorem for periodic difference equations,  $6 \in \mathcal{A}_3$  implies each set  $\mathcal{A}_q$  contains the period of a geometric cycle. From this fact and the fact in Corollary 5, this example has geometric cycles of periods  $6, 4, 8, 16, 20, 24, 28, \dots$

## 5 A converse of Sharkovsky's Theorem for periodic difference equations

A converse to Sharkovsky's Theorem for autonomous maps states that if  $r \prec k$  in the 1-Sharkovsky's ordering, then there is a continuous function defined on a closed interval, which has a point of period  $k$ , but does not have any points of period  $r$  [10]. We depend heavily on this fact to prove the converse of Sharkovsky's Theorem for periodic difference equations.

**Theorem 20 (A converse of Sharkovsky's Theorem for periodic difference equations)** *Given positive integers  $r \geq 1$  and  $p > 1$ , and denote  $\ell := \frac{\text{lcm}(r,p)}{p}$ . There exist a  $p$ -periodic difference equation  $x_{n+1} = f_n(x_n)$  that has a geometric  $r$ -cycle, but no  $r^*$ -cycles for all  $r^* \in \mathcal{A}_q$  for  $q \prec \ell$ .*

**PROOF.** By Elaydi's construction [10], there exists a piecewise linear continuous map  $f(x) : [a, b] \rightarrow [a, b]$  that has a periodic point of minimal period  $\ell$ , but no periodic points of period  $\ell^*$  for all  $\ell^* \prec \ell$  in the 1-Sharkovsky's ordering. Now we define  $f_1, f_2, \dots, f_{p-1}$  to be any bijections on the interval  $[a, b]$ , and define

$$f_0(x) = f_1^{-1} \circ f_2^{-1} \circ \dots \circ f_{p-1}^{-1} \circ f(x).$$

One simple way to do it is by defining  $f_n(x) = x$ ,  $1 \leq n \leq p-1$ , and  $f_0(x) = f(x)$ . Then the  $p$ -fold composition function  $f_{p-1} \circ f_{p-2} \circ \dots \circ f_0(x) = f(x)$ , has no periodic points of any period  $\ell^*$  for any  $\ell^* \prec \ell$ , and consequently the difference equation  $x_{n+1} = f_n(x_n)$  has no  $r^*$ -cycles for all  $r^* \in \mathcal{A}_q \prec \mathcal{A}_\ell$ .

**Corollary 21** *Suppose that  $p = 2^m$  for some positive integer  $m$ , and  $r \geq 1$ . Let  $\ell = \frac{\text{lcm}(r, 2^m)}{2^m}$ . Then there exists a  $p$ -periodic difference equation  $x_{n+1} = f_n(x_n)$  that has a geometric  $r$ -cycle, but no  $r^*$ -cycle for all  $r^* \prec k := \min_{\prec} \mathcal{A}_\ell$  in the 1-Sharkovsky's ordering.*

**PROOF.** Define the set  $\mathcal{A} := \{x : x \prec k \text{ in the 1-Sharkovsky's ordering}\}$ . We show  $\mathcal{A} \subseteq \cup_{q \prec \ell} \mathcal{A}_q$  by investigating the following three cases.

- (i) If  $\ell = 2^j$  for some integer  $j \in \mathbb{N}$  then  $\min_{\prec} \mathcal{A}_{2^j} = k = 2^{j+m}$ , and from the 1-Sharkovsky's ordering and the  $p$ -Sharkovsky's ordering we get  $\mathcal{A} = \cup_{q \prec \ell} \mathcal{A}_q$ .
- (ii) If  $\ell = 2^j(2t+1)$  for some positive integers  $j, t \in \mathbb{Z}^+$  then  $k = 2^{j+m}(2t+1)$ , and as in (i)  $\mathcal{A} = \cup_{q \prec \ell} \mathcal{A}_q$ .
- (iii) Finally, if  $\ell = (2t+1)$  for some  $t \in \mathbb{Z}^+$  then  $k = 2t+1$ , and from the 1-Sharkovsky's ordering and the  $p$ -Sharkovsky's ordering we get  $\mathcal{A} \subset \cup_{q \prec \ell} \mathcal{A}_q$ .

Now the assertion follows from Theorem 20.

**Corollary 22** *Suppose  $p$  is odd and  $r = 2^n q$  where  $q$  is a factor of  $p$  and  $n$  is a positive integer. If there exist a  $p$ -periodic difference equation  $x_{n+1} = f_n(x_n)$  having a geometric  $r$ -cycle but no  $r^*$ -cycle for all  $r^* \prec 2^n q^*$ , where  $q^*$  is the smallest prime factor of  $p$ . Then  $x_{n+1} = f_n(x_n)$  has  $2^{n-1}, 2^{n-2}, \dots, 2^2, 2, 1$  geometric cycles.*

**PROOF.** Observe that  $\ell = \frac{lcm(p,r)}{p} = 2^n$ , so by Sharkovsky's Theorem for periodic difference equations, there corresponds to each  $2^n \prec 2^j$  in the 1-Sharkovsky's ordering a geometric  $r^*$ -cycle for some  $r^* \in \mathcal{A}_{2^j}$ . Now all elements of the set  $\mathcal{A}_{2^j}$  precede  $2^n q^*$  in the Sharkovsky's ordering except  $2^j$ . The given assumption forces the geometric  $r^*$ -cycle to be  $2^j$ ,  $0 \leq j \leq n$ .

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