Linear Almost Periodic Difference Equations*

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Abstract

In this paper, we define the Fourier spectrum of almost periodic sequences and discuss some of its properties. Also we investigate its role on the existence and nonexistence of almost periodic solutions of linear almost periodic difference equations on the semi-group of nonnegative integers. Some of the results provided in this paper can be considered the discrete analog of similar results in almost periodic differential equations.

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1 Introduction

Almost periodic differential equations have been an interesting topic of research for many decades. The Lecture Notes of Fink [22], the memoir of Yoshizawa [39], the books of Hino et al. [28], Levitan and Zhikov [33], Pankov [35], and the references therein contain a great account of this research. On the other hand, little research has been conducted on almost periodic difference equations, though some is found in the work of Blot and Pennequin [6], Corduneanu

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[13], Halanay and Rasvan [24], Hamaya [25, 26, 27], Ignatyev and Ignatyev [29], Pennequin [36], Thanh [38], Zhang [41, 42, 43], and Zhang et al. [44, 45].

In recent years, another line of research has been concerned with difference equations with periodic forcing; see for instance AlSharawi [1], AlSharawi and Angelos [2], AlSharawi et al. [3], Clark and Gross [10], Coleman [11], Cushing and Henson [14, 15, 16], Elaydi and Sacker [17, 18], Franke and Yakubu [19], Henson [20], Jillson [21], Kon [30, 31], Kot and Schaffer [32], and Selgrade and Roberdo [37]. While it is obvious that forcing a periodically fluctuating environment is a very strong constraint, it is more natural to relax this constraint and consider almost periodic fluctuating environment. Here, we are concerned with almost periodic linear difference equations, and in particular, we define the *Fourier spectrum* of an almost periodic sequence and investigate its role in the existence of almost periodic solutions of linear difference equations. We confine ourselves to the semi group of nonnegative integers which distinguish our results from other scattered results in the literature. The spectrum of almost periodic functions has been defined in many ways and used to characterize almost periodic functions as well as the existence of almost periodic solutions of differential equations. For more information on this topic, we refer the reader to Arendt and Schweiker [4], Cartwright [8, 9], Fischer [23] and Naito et al. [34].

In Section 2 of this paper, we give some preliminary results that are used throughout this paper. In Section 3, we define the Fourier spectrum of almost periodic sequences on the nonnegative integers and discuss its characteristics. In Section 4, we investigate existence and nonexistence of almost periodic solutions of the scalar linear difference equation $x_{n+1} = \alpha x_n + f(n)$, and we generalize the results in Section 5 to the vector linear difference equation $X_{n+1} = AX_n + F(n)$. Finally, we generalize the results to the scalar linear equation $x_{n+1} = f(n)x_n + g(n)$ in Section 7. At this point, it is obvious that we have some redundancy in various sections; however, for aesthetic reasons, we begin with a simple setting and proceed to a more general one.

2 Preliminaries

This section is devoted to some preliminary facts required in the sequel. First, let us agree to use the following notations: The sets $\mathbb{Z}, \mathbb{R}, \mathbb{R}^+, \mathbb{C}, \mathbb{Z}^+, \mathbb{N}$ denote integers, real numbers, positive real numbers, complex numbers, positive integers, and non-negative integers respectively. Let $(\mathbb{X}, |\cdot|)$ stand for a Banach space and $\mathcal{C}(\mathbb{X} \times \mathbb{N}, \mathbb{X})$ denote the space of continuous functions on

$\mathbb{X}$.

A function $f(x,n) \in \mathcal{C}(\mathbb{X}\times\mathbb{N},\mathbb{X})$ is said to be almost periodic (AP, for short) in n uniformly for $x \in \mathbb{X}$, if for any $\epsilon > 0$ and compact set K in $\mathbb{X}$, there exists a positive integer $M = M(\epsilon, K)$ such that any set of M consecutive integers $\{N, N+1, ..., N+M\} \subset \mathbb{N}$ contains an integer p for which

$$|f(x,n+p) - f(x,n)| \leq \epsilon, \quad \forall n \in \mathbb{N}.$$

Such a number p is called *an ϵ-translation number* of $f(x,n)$. For simplicity, we call $f(x,n)$ *a discrete AP function,* and we denote the space of such functions by $\mathcal{AP}(\mathbb{X}\times\mathbb{N},\mathbb{X})$. If f is independent of x, we use $\mathcal{AP}(\mathbb{N},\mathbb{X})$ or for short $\mathcal{AP}(\mathbb{N})$. One can use this definition to show that elements of $\mathcal{AP}(\mathbb{N})$ are bounded. Thus, we equip $\mathcal{AP}(\mathbb{N})$ with the supremum norm $||\cdot||_\infty$. The nonautonomous difference equation $x_{n+1} = f(x_n, n)$, $n \in \mathbb{N}$, is called *AP difference equation* or *AP discrete process.* For a function $f \in \mathcal{AP}(\mathbb{N})$ and a nonnegative real number ϵ, we define the set of translation numbers belong to ϵ as

$$T(f,\epsilon) := \{p \in \mathbb{N} : \sup_{n\in\mathbb{N}} |f(n+p) - f(n) < \epsilon\}. \tag{2.1}$$

The theory of AP functions on the space $\mathcal{C}(\mathbb{R},\mathbb{C})$ or $\mathcal{C}(\mathbb{X}\times\mathbb{R},\mathbb{X})$ is very rich with results at our disposal, Besicovitch [5], Bohr [7] and Corduneanu [13]. With a little modification for the space $\mathcal{AP}(\mathbb{X}\times\mathbb{N},\mathbb{X})$, we provide some results that will be referenced at a later point. If there is no confusion, we use $f(x,t)$ when discussing functions in $\mathcal{AP}(\mathbb{X}\times\mathbb{R}^+\cup\{0\},\mathbb{X})$, and we use $f(x,n)$ when discussing functions in $\mathcal{AP}(\mathbb{X}\times\mathbb{N},\mathbb{X})$.

The following theorem can be used to give another equivalent definition of almost periodicity, it is referred to as the *normality condition.*

Theorem 2.1. *$f(x,n)$ is almost periodic in n uniformly for $x \in \mathbb{X}$ if and only if for any sequence $\{h_k\} \subset \mathbb{N}$ there exists a subsequence $\{h'_k\}$ of $\{h_k\}$ and a function $g(x,n)$ such that $f(x,n+h'_k) \to g(x,n)$ uniformly on $K\times\mathbb{N}$ as $k\to\infty$, where K is any compact set in $\mathbb{X}$.*

PROOF: The proof is similar to the proof of Theorem 1.26 in Corduneanu [12], or Theorem 7.2 in Zhang [40]. Thus, we omit it. □

A discrete AP function $f(x,n)$ can have many AP continuous extensions; however, in the sequel, we center our attention on the extension given by

$$f(x,t) = f(x,n) + (t-n)(f(x,n+1) - f(x,n)),\ x \in \mathbb{X}, n \leq t < n+1, n \in \mathbb{N}.$$

Here, we give ourselves the liberty to call this extension "the continuous extension" of $f(x,n)$ to $\mathbb{X}\times\mathbb{R}^+\cup\{0\}$. Let $f(x,n), g(x,n)\in\mathcal{AP}(\mathbb{X}\times\mathbb{N},\mathbb{X})$, $c\in\mathbb{R}$ and $m\in\mathbb{N}$, then

$$cf(x,n),\ f(x,n+m),\ f(x,n)+g(x,n),\ f(x,n)\cdot g(x,n)$$

are elements of $\mathcal{AP}(\mathbb{X},\times\mathbb{N},\mathbb{X})$.

It would be interesting to know when the composition of two functions is AP. The next theorem gives an answer. The proof is similar to the continuous case which can be found in Corduneanu [12] (page 13).

Theorem 2.2. *Let $G:\mathbb{X}^m\to\mathbb{X}$ be a uniformly continuous function on a subset $\mathcal{M}\subseteq\mathbb{X}^m$. If $f_1(n),...,f_m(n)$ are discrete AP functions such that $(f_1(n),...,f_m(n))\in\mathcal{M}$ for all $n\in\mathbb{N}$, then the function $F(n)=G(f_1(n),...,f_m(n))$ is AP.*

Corollary 2.1. *Let $f\in\mathcal{AP}(\mathbb{X}\times\mathbb{N},\mathbb{X})$. e^f is AP and if* $\inf|f(n)|>0$, *then* $\log|f|$ *is AP.*

PROOF: Since f is AP, then $m_1\le f(x,n)\le m_2$ for all $(x,n)\in K\times\mathbb{N}$. Also $|f(x,n)|$ is AP and $|f(x,n)|\le\max\{|m_1|,m_2\}$. Now by Theorem 2.2, e^f and $\log|f|$ are both AP. □

Proposition 2.1. *If $h(n)$ is a discrete AP function, then each of the following holds true.*

(i) *$H(n):=\sum_{j=0}^{n}h(j)$ is AP if and only if it is bounded.*

(ii) *Let $H(0)$ be arbitrary, and define $H(n+1)=H(n)+h(n)$. $H(n)$ is AP if and only if it is bounded.*

(iii) *If $g(n)$ is a summable sequence, i.e., $\sum_{j=0}^{\infty}|g(j)|<\infty$, then*

$$H(n):=\sum_{m=0}^{\infty}g(m)h(n+m)\ \text{is AP.}$$

PROOF: (i) is the discrete version of $H(t)=\int_0^t f(x)dx$ in Corduneanu [12]. (ii) follows from part (i), and (iii) is given in Corduneanu [13] over $\mathbb{Z}$; the proof over $\mathbb{N}$ is similar. □

The next theorem shows that the space of almost periodic functions $\mathcal{AP}(\mathbb{X}\times\mathbb{N},\mathbb{X})$ is closed under the topology of uniform convergence. Its proof follows straightforward from the definitions of convergence and almost periodicity.

Theorem 2.3. *If a sequence $\{f_m(x,n)\}$ of almost periodic functions is uniformly convergent on $\mathbb{X}\times\mathbb{N}$ to $f_0(x,n)$, then f_0 is almost periodic in n uniformly with respect to $x\in\mathbb{X}$.*

Proposition 2.2. *If $f(n)$ is AP, then $|f(n)|$ is AP.*

PROOF: It follows directly from the definition of almost periodicity and the inequality

$$| \, |f(n+p)| - |f(n)| \, | \leq |f(n+p) - f(n)|.$$

The converse of this proposition is not necessarily true; for instance,

$$f(n) = 1, -1, 1, 1, -1, -1, -1, 1, 1, 1, 1, -1, -1, -1, -1, -1, \cdots$$

is not AP, but $|f(n)| = 1 \in \mathcal{AP}(\mathbb{N})$.

Let $f \in \mathcal{AP}(\mathbb{N})$. The translates of f are the elements of the set

$$T(f) = \{f_m : \; f_m(n) = f(m+n), m, n \in \mathbb{N}\}.$$

The set $H(f)$ denotes the closure of $T(f)$ under the topology of uniform convergence and called the hull of f.

Proposition 2.3. *Let $f \in \mathcal{AP}(\mathbb{N})$. If a constant c belongs to the hull of f, then $f = c$.*

PROOF: If $c \in T(f)$, then obviously $f = c$. So let $c \in H(f) \backslash T(f)$ and define $g(n) = f(n) - c$. There exists a sequence $\{\alpha_k\}$ such that $f_{\alpha_k}(n) \to c$ as $k \to \infty$ uniformly in n. Thus g is asymptotic to zero. From the definition of almost periodicity, $g(n)$ is identically zero. Hence, $f(n) = c$. $\square$

3 Fourier spectrum of discrete almost periodic functions

We begin by introducing the Fourier spectrum of an almost periodic sequence.

Definition 3.1. Let $f \in \mathcal{AP}(\mathbb{X} \times \mathbb{N}, \mathbb{X})$ and define

$$M_f(\lambda, x) := \lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k} f(x, j) e^{-\lambda j i}. \tag{3.1}$$

The *exponents* of $f(x, n)$ are the values of λ for which $M_f(\lambda, x) \neq 0$. $M_f(0, x)$ is called the *mean value* of f. When the mean value is independent of x, we write $M_f(\lambda = 0)$, or simply, $M_f(0)$. The set of exponents of f is denoted by $\Lambda(f)$ and called the *Fourier spectrum* of f.

The next theorem establishes the relationship between the exponents of $f(x, n)$ and its continuous extension $f(x, t)$.

Theorem 3.1. *Let $f \in \mathcal{AP}(\mathbb{X} \times \mathbb{N}, \mathbb{X})$, $g(x,t)$ be the continuous extension of f and $\mathbb{X}$ be a separable metric space. Then*

$$M_g(\lambda, x) = \frac{\sin^2(\lambda/2)}{(\lambda/2)^2} M_f(\lambda, x) \ \ where\ \ M_g(\lambda, x) := \lim_{T\to\infty} \frac{1}{T} \int_0^T g(x,t) e^{-i\lambda t} dt.$$

PROOF: Consider the continuous extension of $f(x,n)$,

$$g(x,t) := f(x,j) + (t-j)(f(x,j+1) - f(x,j)),\ j \le t < j+1.$$

$M_g(\lambda, x)$ exists and the set $\Lambda(g) = \{\lambda :\ M_g(\lambda, \cdot) \neq 0\}$ is countable (cf. [39]). Next, the relationship between $\Lambda(g)$ and $\Lambda(f)$ is given.

$$M_g(\lambda, x) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \int_j^{j+1} \left(f(x,j) + (t-j)(f(x,j+1) - f(x,j))\right) e^{-\lambda i t} dt,$$

carrying out the integration and simplifying gives $M_g(\lambda, x)$ equal to

$$\lim_{k\to\infty} \frac{-1}{k\lambda^2} \sum_{j=0}^{k-1} \left[(f(x,j) - (i\lambda+1) f(x,j+1)) e^{-i(j+1)\lambda} + ((i\lambda - 1) f(x,j) + f(x,j+1)) e^{-j\lambda i} \right],$$

which simplifies to

$$M_g(\lambda, x) = \frac{-2}{\lambda^2} (\cos(\lambda) - 1) \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k} e^{-\lambda i j} f(x,j).$$

Hence

$$M_g(\lambda, x) = \frac{-2}{\lambda^2} (\cos(\lambda) - 1) M_f(\lambda, x) = \frac{\sin^2(\lambda/2)}{(\lambda/2)^2} M_f(\lambda, x) \tag{3.2}$$

and the theorem is proved. □

Remark 3.1. *From the proof of Theorem 3.1 and equation (3.2) in particular, the following facts are clear.*

(i) *An exponent of $f(x,t)$ is an exponent of $f(x,n)$; i.e. $\Lambda(f(x,t)) \subseteq \Lambda(f(x,n))$. In fact, $\theta \in \Lambda(f(x,t))$ if and only if the equivalence class $\{\theta + 2q\pi :\ q \in \mathbb{Z}\} \subseteq \Lambda(f(x,n))$.*

(ii) *$\Lambda(f(x,n)) \neq \emptyset$ for any nontrivial AP function $f(x,n)$.*

(iii) *$\Lambda(f)$ is countable.*

Let us illustrate the above results in the following examples. Consider $f(n) = 1, n \in \mathbb{N}$ and the AP continuous extension $f(t) = 1, t \in \mathbb{R}^+ \cup \{0\}$. Then

$$M_g(x, \lambda) = \lim \frac{1}{T} \int_0^T e^{-\lambda t i} dt = \begin{cases} 1 & \text{if } \lambda = 0 \\ 0 & \text{if } \lambda \neq 0, \end{cases}$$

and

$$M_f(x, \lambda) = \lim \frac{1}{k} \sum_{j=0}^{k} e^{-\lambda i j} = \begin{cases} 1 & \text{if } \lambda = 2n\pi \\ 0 & \text{if } \lambda \neq 2n\pi. \end{cases}$$

Recall that an AP function in $\mathcal{AP}(\mathbb{R}, \mathbb{X})$ can either be a trigonometric polynomial or can be approximated uniformly by a sequence of trigonometric polynomials [12], and that is inherited in $\mathcal{AP}(\mathbb{N}, \mathbb{X})$. Thus at least in case of trigonometric polynomials, the series in Definition 3.1 is geometric, which is very simple to evaluate as shown next. The trigonometric polynomial

$$G(n) = \sum_{j=0}^{N} a_j e^{\alpha_j n i}.$$

has

$$\Lambda(G) = \{\alpha_j - 2q\pi,\ q \in \mathbb{Z},\ j = 0, ..., N\}.$$

We caution the reader about equation (3.2), in which the values of λ must be the exponents of $g(x,t)$, while the values of λ in $M_f(\lambda, x)$ can be taken modulo 2π. The smallest additive subgroup of $\mathbb{R}$ containing $\Lambda(f)$ will be denoted by $\text{Mod}(f)$, keeping in mind that the elements of $\Lambda(f)$ are equivalence classes. When a particular exponent of an AP sequence is considered, the principle value will be used; i.e., $\lambda \in \Lambda(f),\ 0 \leq \lambda < 2\pi$. Now, let $f \in \mathcal{AP}(\mathbb{N})$ with exponents $\Lambda(f) = \{\lambda_0, \lambda_1, \ldots\}$. From the continuous extension of $f(n)$ and its associated Fourier series, the Fourier series associated with $f(n)$ is given by $\sum_{j=0}^{\infty} M_f(\lambda_j) e^{-i\lambda_j n}$, where $M_f(\lambda_j)$ are called the *Fourier coefficients.* Furthermore, the elements of the trigonometric sequence $P_m(t) = \sum_{j=0}^{m} A_j e^{-i\lambda_j t}$ that approximate $f(t)$ uniformly approximate $f(n)$ as well. Thus, $P_m(n) = \sum_{j=0}^{m} A_j e^{-i\lambda_j n}$ approximates $f(n)$ uniformly with spectrum contained in $\Lambda(f)$.

Proposition 3.1. *Let $f(n) \in \mathcal{AP}(\mathbb{N})$. If $F(n) = \sum_{j=0}^{n} f(j)$ is AP, then*

(i) $\Lambda(f) \cap \{2q\pi :\ q \in \mathbb{Z}\} = \emptyset$,

(ii) $\Lambda(f) \subset \Lambda(F) \subseteq \Lambda(f) \cup \{2q\pi :\ q \in \mathbb{Z}\}$.

PROOF: Part (i) follows directly from the fact that

$$0=\lim_{k\to\infty}\frac{F(k)}{k}=\lim_{k\to\infty}\frac{1}{k}\sum_{j=0}^{k}f(j).$$

Part (ii), let $\lambda\notin\{2q\pi:\ q\in\mathbb{Z}\}$, then

$$\begin{aligned}\sum_{n=0}^{k}F(n)e^{-\lambda in} &= \sum_{n=0}^{k}e^{-\lambda in}\sum_{j=0}^{n}f(j)\\ &= \sum_{j=0}^{k}f(j)\sum_{n=j}^{k}e^{-\lambda ni}\\ &= \sum_{j=0}^{k}f(j)\left(\frac{e^{-\lambda ij}-e^{-\lambda(k+1)i}}{1-e^{-\lambda i}}\right)\end{aligned}$$

Divide by k and let $k\to\infty$. Then

$$\lim\frac{1}{k}\sum_{n=0}^{k}F(n)e^{-\lambda in}=\frac{1}{1-e^{-\lambda i}}\lim\frac{1}{k}\sum_{j=0}^{k}f(j)e^{-\lambda ij},$$

which together with (i) implies

$$\Lambda(f)\subset\Lambda(F)\subseteq\Lambda(f)\cup\{2q\pi:\ q\in\mathbb{Z}\}.$$

□

Here, it is worth mentioning that $\Lambda(F)=\Lambda(f)$ or $\Lambda(F)=\Lambda(f)\cup\{2q\pi:\ q\in\mathbb{Z}\}$ as illustrated by this example. Let $f(n)=a_1e^{\lambda_1 in}+a_2e^{\lambda_2 in}$, where a_1 and a_2 are chosen so that $\Lambda(f)=\{\lambda_1+2q\pi,\ \lambda_2+2q\pi:\ q\in\mathbb{Z}\}$. Also, assume $\lambda_1,\lambda_2\neq 0 \bmod 2\pi$. In this case $F(n)=\sum_{j=0}^{n}f(j)$ is AP. Furthermore,

$$\Lambda(F)=\begin{cases}\Lambda(f) & \text{if}\quad a_2\neq\frac{-a_1(1-e^{\lambda_2 i})}{1-e^{\lambda_1 i}}\\ \Lambda(f)\cup\{2q\pi:\ q\in\mathbb{Z}\} & \text{if}\quad a_2=\frac{-a_1(1-e^{\lambda_2 i})}{1-e^{\lambda_1 i}}.\end{cases}$$

This example and Proposition 3.1 imply that $F^*(k)=\sum_{n=0}^{k}F(n)$ cannot be AP.

In case of a continuous AP function $g\in\mathcal{AP}(\mathbb{R},\mathbb{C})$ with Fourier series $\sum_{j=0}^{\infty}M_g(\lambda_j)e^{-i\lambda_j t}$, Parseval's equation states that

$$\sum_{j=0}^{\infty}|M_g(\lambda_j)|^2=\lim\frac{1}{T}\int_0^T|g(t)|^2dt; \tag{3.3}$$

that is, the sequence $M_g(\lambda_j)$ is a square summable sequence. This fact, together with equation (2) imply

$$\sum_{j=0}^{\infty}|M_g(\lambda_j)|^2=\sum_{j=0}^{\infty}\frac{4}{\lambda^4}(1-\cos\lambda_j)^2|M_f(\lambda_j)|^2.$$

It is interesting to know a sufficient condition on the spectrum of f that makes $F(n)=\sum_{j=0}^{n}f(j)$ AP. The next example and proposition give the answer.

Example 3.1. *Define the sequence*

$$f_m(n) = \sum_{j=0}^{m} M_{f_m}(\lambda_j) e^{\lambda_j in}, \quad \text{where } \lambda_j = \frac{1}{2^j} \quad \text{and } M_{f_m}(\lambda_j) = 1 - e^{\lambda_j i}.$$

$f_m(n) \to f_\infty(n) = \sum_{j=0}^{\infty}(1 - e^{\lambda_j i})e^{\lambda_j in}$ *as* $m \to \infty$*. From Parseval's equation and equation (3.2),* $M_{f_m}(\lambda_j)$ *is square summable.*

$$\sum_{j=0}^{m} |M_{f_m}(\lambda_j)|^2 = \sum_{j=0}^{m} |1 - e^{\lambda_j i}|^2 = \sum_{j=0}^{m} \sin^2(\lambda_j/2) \leq \sum_{j=0}^{m} \frac{1}{2^{2j+2}}$$

and

$$\sum_{j=0}^{m} |M_g(\lambda_j)|^2 = \sum_{j=0}^{m} \frac{\sin^4(\lambda_j/2)}{(\lambda_j/2)^4} \sin^2(\lambda_j/2) \leq \sum_{j=0}^{m} \frac{1}{2^{2j+2}}.$$

Thus, $f_m \to f_\infty$ *uniformly, and consequently,* f_∞ *is AP with* $\Lambda(f_\infty) = \{\frac{1}{2^j} + 2q\pi : \; j \in \mathbb{N}, \; q \in \mathbb{Z}\}$*. Now*

$$F(n) = \sum_{j=0}^{n} f_\infty(j) = \sum_{k=0}^{\infty} M_{f_m}(\lambda_k) \frac{1 - e^{(n+1)\lambda_k i}}{1 - e^{\lambda_k i}} = \lim_{m \to \infty} \sum_{k=0}^{m} (1 - e^{(n+1)\lambda_k i})$$

and as $m \to \infty$*, it is not AP. Observe in this example that* $\inf \Lambda(f_\infty) = 0$*, which leads us to the following question. Is it possible to have* $\inf \Lambda(f_\infty) = 0$ *and obtain an AP sequence* $F(n) = \sum_{j=0}^{n} f_\infty(j)$*? Indeed, the answer is positive as seen by defining* $M_{f_m}(\lambda_j)$ *above to be* $\lambda_j(1 - e^{\lambda_j i})$*.* f_∞ *is almost periodic and*

$$F(n) = \sum_{j=0}^{n} f_\infty(j) = \sum_{j=0}^{\infty} \frac{1}{2^j}(1 - e^{(n+1)\lambda_j i})$$

is AP as well.

Proposition 3.2. *Let* $f \in \mathcal{AP}(\mathbb{N})$*. If* $\inf\{|\lambda| : \; \lambda \in \Lambda(f)\} > 0$*, then* $F(n) = \sum_{j=0}^{n} f(j)$ *is AP.*

PROOF: Since $g(t) = f(j) + (t - j)(f(j + 1) - f(j)), j \leq t < j + 1, j \in \mathbb{N}$ is AP and $g(n)$ is discrete AP, we apply Favard's Theory on $g(t)$. From Theorem 3.1 and Remark 3.1, $\inf\{|\lambda| : \; \lambda \in \Lambda(f)\} > 0$ implies $\inf\{|\lambda| : \; \lambda \in \Lambda(g)\} > 0$. Thus, $G(t) = \int_0^t g(s)ds$ is AP, and consequently, $G(n) = \int_0^n g(s)ds$ is discrete AP. Now,

$$G(n) = \sum_{j=0}^{n-1} \int_j^{j+1} [f(j) + (s - j)(f(j + 1) - f(j))] \, ds = \sum_{j=0}^{n-1} f(j) - \frac{1}{2}(f(n) + f(0)).$$

Since $G(n) + \frac{1}{2}(f(n) + f(0))$ belongs to $\mathcal{AP}(\mathbb{N})$, then $F(n)$ does as well. □

Proposition 3.3. *Let $f \in \mathcal{AP}(\mathbb{X} \times \mathbb{N}, \mathbb{X})$. For each $g \in H(f)$, $\Lambda(g) = \Lambda(f)$. Furthermore, $M_{g_1}(\lambda = 0) = M_{g_2}(\lambda = 0)$ for all $g_1, g_2 \in H(f)$.*

PROOF: If $g \in T(f)$, then $g = f_m$ for some $m \in \mathbb{N}$; i.e. $g(x,n) = f(x, n+m)$. Now

$$\sum_{j=0}^{k} g(x,j)e^{-\lambda ij} = \sum_{j=0}^{k} f(x, j+m)e^{-\lambda ii} = \sum_{s=m}^{k+m} f(x,s)e^{-\lambda(s-m)i} = e^{\lambda mi} \sum_{s=m}^{k+m} f(x,s)e^{-\lambda is}.$$

Thus $\Lambda(f) = \Lambda(g)$. If $g \notin T(f)$, but $g = \lim f_{t_m}$ for some subsequence $\{t_m\}$ of $\mathbb{N}$, for each $\epsilon > 0$ there exists $M := M(x, \epsilon) \in \mathbb{N}$ such that

$$|f_{t_m} - g| < \epsilon, \quad \forall m > M.$$

Now,

$$\left| \frac{1}{k} \sum_{j=0}^{k} g e^{-\lambda ij} - \frac{1}{k} \sum_{j=0}^{k} f_{t_m} e^{-\lambda ij} \right| = \frac{1}{k} \left| \sum_{j=0}^{k} (g - f_{t_m}) e^{-\lambda ij} \right| < \epsilon, \ \forall m > M. \tag{3.4}$$

Let λ be an exponent of f; it is an exponent of f_{t_m} for all m. If λ is not an exponent of g, then we obtain

$$\lim \frac{1}{k} \sum_{j=0}^{k} f_{t_m} e^{-\lambda ij} \neq 0 \quad \text{and} \quad \lim \frac{1}{k} \sum_{j=0}^{k} g e^{-\lambda ij} = 0,$$

and that contradicts inequality (3.4) for sufficiently large m. Similarly, if we consider λ to be an exponent of g but not an exponent of f, hence $\Lambda(f) = \Lambda(g)$. The rest of the assertion follows from the definition of the mean. □

Let $f(n) \in \mathcal{AP}(\mathbb{N})$ and define $F(n) = \sum_{j=0}^{n} |f(n)|$. $F(n)$ is a monotonic sequence if it is bounded, then it must converge. Therefore, $F(n)$ cannot be an AP sequence. Furthermore, $0 \in \Lambda(|f|)$. To clarify this fact, let $\lambda \in \Lambda(f)$,

$$\lim \frac{1}{k} \sum_{j=0}^{k} f(j)e^{-\lambda ij} \neq 0$$

and

$$0 < \lim \frac{1}{k} \Big| \sum_{j=0}^{k} f(j)e^{-\lambda ij} \Big| \leq \lim \frac{1}{k} \sum_{j=0}^{k} |f(j)|.$$

Thus, 0 is an exponent of $|f|$.

The next proposition establishes a relationship between the modules and translation numbers of AP functions.

Proposition 3.4. *Let f and $g \in \mathcal{AP}(\mathbb{N})$. $Mod(g) \subseteq Mod(f)$ if and only if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $T(f,\delta) \subseteq T(g,\epsilon)$.*

PROOF: Apply Theorem 4.5 in Fink [22] to the continuous extensions of $f(n)$ and $g(n)$. The rest follows from the definition of translation sets and modules. □

We end this section with the following simple fact.

Proposition 3.5. *Let $g_1, g_2 \in \mathcal{AP}(\mathbb{N})$ and a, b be nonzero real numbers. Each of the following holds true.*

(i) $\Lambda(ag_1 + bg_2) \subseteq \Lambda(g_1) \cup \Lambda(g_2)$

(ii) $(\Lambda(g_1) \setminus \Lambda(g_2)) \cup (\Lambda(g_2) \setminus \Lambda(g_1)) \subseteq \Lambda(ag_1 + bg_2)$

(iii) $\mathrm{Mod}(g_1 g_2) \subseteq \mathrm{Mod}(g_1) \cap \mathrm{Mod}(g_2)$.

PROOF: Let $\lambda \in \mathbb{R}$. The fact that

$$\frac{1}{n}\sum_{j=0}^{n}(ag_1(j) + bg_2(j))e^{-\lambda ij} = \frac{a}{n}\sum_{j=0}^{n} g_1(j)e^{-\lambda ij} + \frac{b}{n}\sum_{j=0}^{n} g_2(j)e^{-\lambda ij}$$

clarifies (i) and (ii). Now,

$$|g_1(n+p)g_2(n+p) - g_1(n)g_2(n)| \leq |g_1(n+p)|\,|g_2(n+p) - g_2(n)| + |g_2(n)|\,|g_1(n+p) - g(n)|$$

together with Proposition 3.4 complete the proof of part (iii). □

4 The difference equation $x_{n+1} = \alpha x_n + f(n)$

First, let us consider the simple homogeneous equation $x_{n+1} = \alpha x_n$, where $\alpha \in \mathbb{C}$ is fixed. Clearly, if $|\alpha| < 1$ or $|\alpha| > 1$, then $|x_n| \to 0$ or $|x_n| \to \infty$ consequently, and no nontrivial AP solution exists. However, if $|\alpha| = 1$, then every solution is AP and given by $x_n = e^{in\theta}x_0$ for some $\theta \in \mathbb{R}$.

Next, consider the nonhomogeneous scalar difference equation

$$x_{n+1} = \alpha x_n + f(n), \tag{4.1}$$

where $f \in \mathcal{AP}(\mathbb{N})$, $f \not\equiv 0$, and $\alpha \in \mathbb{C}$. Let $|\alpha| = 1$. Since every solution of $x_{n+1} = \alpha x_n$ is AP, the existence of one AP solution of equation (4.1) guarantees that every solution is AP. The next two results characterize AP solutions of equation (4.1). First, denote by $\mathbb{S}$ the unit circle; i.e. $\mathbb{S} := \{x \in \mathbb{X} : \ |x| = 1\}$, and denote by $\mathbb{S}_f$ the set $\{e^{i\lambda} : \ \lambda \in \Lambda(f)\}$.

Proposition 4.1. *If $\alpha \in \mathbb{S}_f$, then equation (4.1) has no AP solution.*

PROOF: Let $\alpha \in \mathbb{S}_f$ and suppose y_n is an AP solution. Then $y_{n+1} = \alpha y_n + f(n)$, multiply both sides by $e^{-\lambda ni}$ for some $\lambda \in \Lambda(f)$ and sum over n as follows:

$$\sum_{n=0}^{k} y_{n+1}e^{-\lambda ni} = \alpha \sum_{n=0}^{k} y_n e^{-\lambda ni} + \sum_{n=0}^{k} f(n)e^{-\lambda ni}.$$

Divide by k and rewrite the equation as

$$(e^{\lambda i} - \alpha)\frac{1}{k}\sum_{n=0}^{k} y_n e^{-\lambda ni} = \frac{y_0}{k}e^{\lambda i} - \frac{y_{k+1}}{k}e^{-\lambda ik} + \frac{1}{k}\sum_{n=0}^{k} f(n)e^{-\lambda ni}. \tag{4.2}$$

Since $\alpha \in \mathbb{S}_f$, the left hand side of the equation is zero. Also, since y_n is AP by assumption, then it is bounded. As $k \to \infty$, the first two terms of the right hand side are zeros, but the last term in the equation is not zero, which is a contradiction. □

Parts (i) and (ii) of the next theorem are discussed (more or less) in Corduneanu [13], Halanay and Rasvan [24], and Pennequin [36], where $f \in \mathcal{AP}(\mathbb{Z}, \mathbb{X})$. Here, we are confining ourselves with the space $\mathcal{AP}(\mathbb{N}, \mathbb{X})$ or $\mathcal{AP}(\mathbb{X} \times \mathbb{N}, \mathbb{X})$, and we focus on the role of the spectrum.

Theorem 4.1. *Let $\alpha \notin \mathbb{S}_f$, each of the following holds true for Eq. (4.1).*

(i) *If $|\alpha| < 1$, then each solution is bounded, and there exists a unique AP solution. Furthermore, the AP solution has the same exponents as f and its globally asymptotically stable.*

(ii) *If $|\alpha| > 1$, then there exists a unique AP solution with same exponents as f.*

(iii) *If $\alpha = e^{\theta i}$ and $\inf_{\lambda \in \Lambda(f)} |\lambda + \theta| > 0$, then every solution is AP. Furthermore, the exponents of any solution are contained in $\Lambda(f) \cup \{\theta + 2q\pi : \ q \in \mathbb{Z}\}$.*

PROOF:

(i) Unlike the proof on $\mathbb{Z}$, a constructive proof over $\mathbb{N}$ is more intricate; however, we give a short proof using fixed point theory on the Banach space $\mathcal{AP}(\mathbb{N}, \mathbb{X})$ endowed with the supremum norm $||\cdot||_\infty$. First, let us show that every solution is bounded. Let $|\alpha| < 1$ and $z_0 \in \mathbb{X}$,

$$\begin{aligned} z_n =& \ \alpha^n z_0 + \alpha^n \textstyle\sum_{j=0}^{n-1} \alpha^{-j-1} f(j) \\ |z_n| \leq& \ |\alpha|^n |z_0| + \textstyle\sum_{j=0}^{n-1} |\alpha|^{n-j-1} |f(j)| \\ |z_n| \leq& \ \max\{|z_0|, ||f||_\infty\} \textstyle\sum_{j=0}^{n} |\alpha|^j \\ \leq& \ \tfrac{1}{1-|\alpha|} \max\{|z_0|, ||f||_\infty\}, \end{aligned}$$

so every solution z_n is bounded. Next, define the operator

$$T: \mathcal{AP}(\mathbb{N}) \to \mathcal{AP}(\mathbb{N})$$

as $T(X) = U$, where $X = \{x_n\} \in \mathcal{AP}(\mathbb{N})$ and $U = \{u_n\}$ such that $u_{n+1} = \alpha x_n + f(n)$. Observe that we need to define u_0. Thus, we extend X and f to be defined at $n = -1$ as follows: Since there exists trigonometric polynomials $P_{f,m}(n) = \sum_{j=0}^{m} A_j e^{-\lambda_j in}$ and $Q_{X,m}(n) = \sum_{j=0}^{m} B_j e^{-\beta_j i} n$ that approximate f and X respectively, then we define $f(-1) = P_{f,\infty}(-1)$ and $x_{-1} = Q_{X,\infty}(-1)$. It is straightforward to check that f and X became AP on $\mathbb{N} \cup \{-1\}$. Now, $U \in \mathcal{AP}(\mathbb{N})$ and T is well-defined. Next, let $Y = \{y_n\}$, $V = \{v_n\}\}$ such that $T(Y) = V$.

$$\begin{aligned} ||T(X) - T(Y)||_\infty &= ||U - V||_\infty \\ &= \sup\{|u_n - v_n| : n \in \mathbb{N}\} \\ &= \sup\{\alpha|x_{n-1} - y_{n-1}| : n \in \mathbb{N}\} \\ &= |\alpha| \sup\{|x_n - y_n| : n \in \mathbb{N}\} \\ &= |\alpha| \, ||X - Y||_\infty. \end{aligned}$$

Notice that removing a point from an AP sequence does not alter its supremum norm. Thus, T is a contraction, thereby having a fixed point, which is the required AP solution. To show its uniqueness, assume there exists another AP solution, say y_n. Then subtract the two solutions

$$z_n = \alpha^n z_0 + \alpha^n \sum_{j=0}^{n-1} \alpha^{-j-1} f(j) \quad \text{and} \quad y_n = \alpha^n y_0 + \alpha^n \sum_{j=0}^{n-1} \alpha^{-j-1} f(j)$$

to obtain

$$z_n - y_n = \alpha^n (z_0 - y_0).$$

Since the L.H.S. is AP, the R.H.S. must be AP, which is obviously not possible unless $z_0 = y_0$. Also, from the same equation, z_n is globally asymptotically stable. Finally, show that z_n has the same exponents as f. Follow the procedure of Proposition 4.1 to obtain equation (4.2), take the limit as $k \to \infty$ to obtain

$$(e^{\lambda i} - \alpha) \lim \frac{1}{k} \sum_{n=0}^{k} z_n e^{-\lambda ni} = \lim \frac{1}{k} \sum_{n=0}^{k} f(n) e^{-\lambda ni}, \tag{4.3}$$

and from which $\Lambda(z_n) = \Lambda(f(n))$.

(ii) Let $|\alpha| > 1$. First show the existence of a unique bounded solution. Any solution can be written as

$$z_n = \alpha^n \left(z_0 + \sum_{j=0}^{n-1} \frac{f(j)}{\alpha^{j+1}} \right),$$

from which it is clear that $z_0 = \sum_{j=0}^{\infty} \frac{-f(j)}{\alpha^{j+1}}$ provides the unique bounded solution. Indeed, $z_n = -\alpha^n \sum_{j=n}^{\infty} f(j)/\alpha^{j+1}$ implies

$$|z_n| \leq ||f||_\infty \sum_{j=n}^{\infty} |\alpha|^{n-j-1} = \frac{||f||_\infty}{|\alpha| - 1}. \tag{4.4}$$

Now for this particular solution,

$$\begin{aligned} z_{n+p+1} - z_{n+1} = &\ \alpha(z_{n+p} - z_n) + f(n+p) - f(n) \\ \alpha(z_{n+p} - z_n) = &\ (z_{n+p+1} - z_{n+1}) - (f(n+p) - f(n)) \\ |\alpha||(z_{n+p} - z_n)| \leq &\ |(z_{n+p+1} - z_{n+1})| + |(f(n+p) - f(n))| \\ |\alpha| \sup_n |z_{n+p} - z_n| \leq &\ \sup_n |z_{n+p+1} - z_{n+1}| + \sup_n |f(n+p) - f(n)|, \end{aligned}$$

and from which

$$\sup_n |z_{n+p} - z_n| \leq \frac{1}{|\alpha| - 1} \sup_n |f(n+p) - f(n)|. \tag{4.5}$$

Thus, an $\epsilon/(|\alpha| - 1)$ translation number of $f(n)$ is an ϵ-translation number of z_n. Finally, $\Lambda(z_n) = \Lambda(f)$ follows as in (i).

(iii) Consider $\alpha = e^{i\theta}$, any solution y_n is given by

$$y_n = e^{n\theta i} y_0 + e^{\theta(n-1)i} \sum_{j=0}^{n-1} e^{-j\theta i} f(j). \tag{4.6}$$

If λ is an exponent of f, then $\theta + \lambda$ is an exponent of $g(n) = e^{-n\theta i} f(n)$. Since $\inf |\theta + \lambda| > 0$, by Proposition 3.2, $G(n) = \sum_{j=0}^{n-1} g(j)$ is AP; consequently, y_n is AP for any choice of the initial condition y_0. By an argument similar to that in equation (4.6)

$$(e^{\lambda i} - e^{i\theta}) \lim_{k \to \infty} \frac{1}{k} \sum_{n=0}^{k} y_n e^{-\lambda n i} = \lim_{k \to \infty} \frac{1}{k} \sum_{n=0}^{k} f(n) e^{-\lambda n i},$$

which implies

$$\Lambda(f) \subset \Lambda(y_n) \subseteq \Lambda(f) \cup \{\theta + 2q\pi : \ q \in \mathbb{Z}\}.$$

Hence, the result is proved.

□

We close this section with an interesting consequence of Theorem 4.1.

Corollary 4.1. *Let $f \in \mathcal{AP}(\mathbb{N})$ with exponents $\Lambda(f) = \{\lambda_1, \lambda_2, \cdots\}$. If $\alpha \notin \mathbb{S}$, then*

$$\sum_{j=0}^{\infty} \frac{|M_f(\lambda_j)|^2}{|e^{\lambda_j i} - \alpha|^2}$$

converges.

PROOF: Since $\alpha \notin \mathbb{S}$, then by Theorem 4.1, Eq. (4.1) has a unique AP solution x_n with the same exponents as f. Furthermore, from Eq. (4.3), the Fourier coefficients $M_{x_n}(\lambda_j)$ are given by $M_f(\lambda_j)/(e^{\lambda_j i} - \alpha)$. Now, Parseval's equality completes the proof. □

5 The equation $X_{n+1} = AX_n + F(n)$

Let us begin by considering the homogenous difference equation

$$X_{n+1} = AX_n, \tag{5.1}$$

where A is an $m \times m$ constant matrix. Write $A = SBS^{-1}$ where S is nonsingular matrix, and B is a lower triangular matrix with the same spectrum as A, say $\sigma(A) = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ in the main diagonal. Let $S^{-1}X_n = Y_n$, $Y_{n+1} = BY_n$. Separate the equations in this system from the top down. From the discussion following equation (4.1), it is straightforward to conclude that $\mathbb{S} \cap \sigma(A) = \emptyset$ implies no nontrivial AP solutions. The case $\mathbb{S} \cap \sigma(A) \neq \emptyset$ is more interesting. Assume the existence of an eigenvalue of B inside $\mathbb{S}$, say $\alpha = e^{i\theta} \in \mathbb{S}$, and let X_0 be the eigenvector belonging to α, $AX_0 = \alpha X_0$. Thus, we obtain

$$X_n = A^n X_0 = A^{n-1} \alpha X_0 = \alpha A^{n-1} X_0 = \alpha^n X_0 = e^{in\theta} X_0.$$

We summarize this discussion in the next result.

Proposition 5.1. *Consider the difference equation $X_{n+1} = AX_n$, and suppose the matrix A has a spectrum $\sigma(A)$. Each of the following holds true.*

(i) *If $\sigma(A) \cap \mathbb{S} = \emptyset$, then there is no nontrivial AP solution.*

(ii) *If $\sigma(A) \cap \mathbb{S} \neq \emptyset$, then there exists an infinite number of nontrivial AP solutions with Fourier spectrum $\{\theta + 2q\pi\}$, where $\alpha_j = e^{i\theta}$ for some $j = 1, ..., m$.*

Now, we are ready to discuss the nonhomogeneous vector equation

$$X_{n+1} = AX_n + F(n), \; n \in \mathbb{N}, \tag{5.2}$$

where $F(n) = (f_1(n), f_2(n), ..., f_m(n))^t$ is not identically zero, each component $f_j \in \mathcal{AP}(\mathbb{N})$, and A is an $m \times m$ matrix as before with eigenvalues $\alpha_1, \alpha_2, ..., \alpha_m$. Again here, if $\alpha_j \in \mathbb{S}$ for some j, we write $\alpha_j = e^{i\theta_j}$ for some principle value $0 \leq \theta_j < 2\pi$. Corduneanu [13], and Pennequin [36] proved that equation (5.2) has an AP solution if and only if, it has a bounded solution on the set of integers. Again, here we are concerned with the set of nonnegative integers, and prefer to treat this equation using the exponents approach. As before, write $A = SBS^{-1}$ and let $y_n = S^{-1}x_n$, then

$$y_{n+1} = By_n + F^*(n),$$

where $F^*(n) = S^{-1}F(n)$. By separating the equations in this system from the top down, the first equation is in the form

$$z_{n+1} = \alpha_1 z_n + f_1^*(n). \tag{5.3}$$

Now from this fact, Theorem 4.1 and Proposition 5.1, we obtain the next result. First, let us agree to denote the region of the complex plain inside the unit circle by $\mathbb{S}_I$ and the region outside the unit circle by $\mathbb{S}_O$. Thus $\mathbb{C} = \mathbb{S}_I \cup \mathbb{S} \cup \mathbb{S}_O$ and $\mathbb{S}_I, \mathbb{S}, \mathbb{S}_O$ are mutually disjoint.

Theorem 5.1. *Consider equation (5.2) and denote the spectrum of A by $\sigma(A)$. Then each of the following holds true.*

(i) *If $\sigma(A) \subset \mathbb{S}_I$, then each solution is bounded, and there exists a unique AP solution. Furthermore, the AP solution has the same exponents as F and it is GAS.*

(ii) *If $\sigma(A) \subset \mathbb{S}_O$, then there exists a unique AP solution. Furthermore, this AP solution has the same spectrum as F, and all other solutions are unbounded.*

(iii) *If $\sigma(A) \subset \mathbb{S}$ and $\inf_{\lambda \in \Lambda(F^*)} |\theta_j + \lambda| > 0$ for all $1 \leq j \leq m$, then every solution is AP. Furthermore, any AP solution has exponents contained in $\Lambda(F) \cup \{\theta_j + 2q\pi : \; q \in \mathbb{Z}, 1 \leq j \leq m\}$.*

In the same way as in Theorem 5.1 and the preceding discussion, it is possible to tackle the following cases:

- $\sigma(A) \subset \mathbb{S} \cup \mathbb{S}_I$, and $\sigma(A) \cap \mathbb{S} \neq \emptyset, \sigma(A) \cap \mathbb{S}_I \neq \emptyset$

- $\sigma(A) \subset \mathbb{S} \cup \mathbb{S}_O$, and $\sigma(A) \cap \mathbb{S} \neq \emptyset, \sigma(A) \cap \mathbb{S}_O \neq \emptyset$
- $\sigma(A) \subset \mathbb{S}_O \cup \mathbb{S}_I$, and $\sigma(A) \cap \mathbb{S}_O \neq \emptyset, \sigma(A) \cap \mathbb{S}_I \neq \emptyset$
- $\sigma(A) \cap \mathbb{S} \neq \emptyset, \sigma(A) \cap \mathbb{S}_I \neq \emptyset$, and $\sigma(A) \cap \mathbb{S}_O \neq \emptyset$.

However, because these cases are simple and awkward to write, the details are omitted.

6 The equation $x_{n+1} = f(n)x_n$

Consider the scalar equation

$$x_{n+1} = f(n)x_n, \tag{6.1}$$

where $f \in \mathcal{AP}(\mathbb{N}, \mathbb{X})$ is nonconstant. Clearly, any solution is given by $x_n = \prod_{j=0}^{n-1} f(j)x_0$ and our concern is with nontrivial AP solutions. If $f(j) = 0$ for some $j \in \mathbb{N}$, then every solution is eventually zero, and consequently, no nontrivial AP solution exists. If $\inf f(j) \geq 1$, then nontrivial solutions are unbounded, and consequently no AP solution exists. Also, if $\sup f(j) \leq 1$, then every nontrivial solution is asymptotic to zero, and consequently, cannot be AP. From this discussion, it is legitimate to proceed with the assumptions $\inf |f| < 1, \sup |f| > 1$ and $f(j) \neq 0,\ \forall j \in \mathbb{N}$.

The geometric mean of an AP sequence $f(n)$ is defined as

$$M_f^* = \lim_{k\to\infty} \left(\prod_{j=0}^{k} |f(j)| \right)^{\frac{1}{k}}.$$

Observe that if $\inf |f(j)| > 0$, then $\ln |f(n)|$ is AP; thus, $\lim \frac{1}{k} \sum_{j=0}^{k} \ln |f(j)|$ exists. Now, $\ln(M_f^*) = M_{\ln|f|}(\lambda = 0)$ and hence M_f^* exists whenever $\inf |f(j)| > 0$.

It is well known that

$$\left(\prod_{j=0}^{n} |f(j)| \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{j=0}^{n} |f(j)| \text{ for all } n \in \mathbb{Z}^+,$$

which implies $M_f^* \leq M_{|f|}(\lambda = 0)$. Another simple but important fact is

$$M_f^* = \lim_{k\to\infty} \left(\prod_{j=0}^{k} |f(j)| \right)^{\frac{1}{k}} = \lim_{k\to\infty} \left(\prod_{j=m}^{k+m} |f(j)| \right)^{\frac{1}{k}} \tag{6.2}$$

uniformly in m. This follows from the fact that $\inf |f(j)| > 0$ implies $\ln |f(n)|$ is AP and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=m}^{n+m} \ln |f(j)| = M_{\ln |f|}(0).$$

This discussion is summarized in the following proposition.

Proposition 6.1. *Let $f \in \mathcal{AP}(\mathbb{N}, \mathbb{X})$ such that $\inf |f(j)| > 0$. Each of the following holds true.*

(i) *M_f^* is well defined.*

(ii) $M_f^* \leq M_{|f|}(0)$.

(iii) $M_f^* = \lim_{k\to\infty} \left(\prod_{j=0}^{k} |f(j)|\right)^{\frac{1}{k}} = \lim_{k\to\infty} \left(\prod_{j=m}^{k+m} |f(j)|\right)^{\frac{1}{k}}$ *for any nonnegative integer m.*

For our convenience, we define $f_{n,m} := \prod_{j=m}^{n} f(j),\ \forall n \geq m$ and $F(n) = f_{n,0}$. Also, by convention, we define $f_{n,m} = 1$ whenever $m > n$. The next result will be referenced at a later point.

Lemma 6.1. *Let $f \in \mathcal{AP}(\mathbb{N}, \mathbb{X})$ such that $\inf |f(j)| > 0$. Each of the following holds true.*

(i) *If $M_f^* > 1$, then $\sum_{j=0}^{\infty} f_{j,0}^{-1}$ converges absolutely.*

(ii) *If $M_f^* < 1$, then $\sum_{j=0}^{\infty} f_{j,0}$ converges absolutely.*

(iii) *If $M_f^* < 1$, then $\lim_{n\to\infty} \sum_{j=0}^{n} |f_{n,j+1}| < \infty$.*

(iv) *If $M_{|f|}(0) < 1$, then $\lim_{n\to\infty} \sum_{j=0}^{n} |f_{n,j+1}| < \infty$.*

PROOF: Parts (i) and (ii) follow directly from the root test. So, it remains to show (iii) and (iv). To show (iii), let $M_f^* = B < 1$, from Eq. 6.2, there exists $N_0 \in \mathbb{N}$ such that

$$\prod_{j=m}^{n+m} |f(j)| \leq \left(\frac{B+1}{2}\right)^n, \ \forall n \geq N_0. \tag{6.3}$$

For sufficiently large n,

$$\sum_{j=0}^{n} |f_{n,j+1}| = \sum_{j=0}^{N_0-1} |f_{n,j+1}| + \sum_{j=N_0}^{n-N_0-1} |f_{n,j+1}| + \sum_{j=n-N_0}^{n} |f_{n,j+1}|. \tag{6.4}$$

From inequality (6.3), $\sum_{j=0}^{N_0-1}|f_{n,j+1}| \;\to\; 0$ as $n\to\infty$, Also, from part (iii) in Proposition 6.1 and for sufficiently large n, we obtain

$$\sum_{j=N_0}^{n-N_0-1}|f_{n,j+1}| = \prod_{j=N_0+1}^{n}|f(j)| + \prod_{j=N_0+2}^{n}|f(j)| + ... + \prod_{j=n-N_0-1}^{n}|f(j)| < \frac{2}{1-B}\left(\frac{B+1}{2}\right)^{N_0+1}$$

Now, let $M_f^* := B < 1$; we obtain

$$\sum_{j=n-N_0}^{n}|f_{n,j+1}| \leq 1 + M + M^2 + ... + M^{N_0+1} = \begin{cases} N_0+2 & \text{if } M=1 \\ \frac{1-M^{N_0+2}}{1-M} & \text{if } M\neq 1. \end{cases}$$

Thus, the R.H.S. of Eq. (6.4) is finite and the proof of (iii) is complete. Finally, (iv) follows directly from (iii) and (ii) in Proposition 6.1. □

Next, we utilize the geometric mean to obtain the following two results.

Proposition 6.2. *Let $f\in\mathcal{AP}(\mathbb{N},\mathbb{X})$ such that* $\inf|f(j)|\neq 0$. *If* $0\in\Lambda(\ln|f|)$, *then* $x_{n+1} = f(n)x_n$ *has no nontrivial AP solution.*

PROOF: Since $0\in\Lambda(\ln|f(j)|)$, then $M_{\ln|f|}(0)\neq 0$, and consequently, $M_f^*\neq 1$. Now, we use the root test of convergence to complete the proof. If $M_f^*>1$, then $f_{n,0}$ is unbounded, and if $M_f^*<1$, then $f_{n,0}$ converges to 0. In both cases, $f_{n,0}=\prod_{j=0}^{n}f(j)$ cannot be a nontrivial AP sequence. □

The more challenging case is when $M_f^*=1$. For instance, if $f(n)=e^{2ni}$, then $M_f^*=1$, but $F(n)=\prod_{j=0}^{n}f(j)=e^{in(n+1)}$ is not AP. On the other hand, consider $f(j)=\frac{1}{2}$ if j is even and $f(j)=2$ if j is odd. Then $M_f^*=1$ and $F(n)=\prod_{j=0}^{n}f(j)$ is AP. It would be interesting to construct an example in which $M_f^*=1$ and $F(n)=\prod_{j=0}^{n}f(j)$ is not periodic, but rather an AP. However, our attempts were unsuccessful.

Proposition 6.3. *Let $f\in\mathcal{AP}(\mathbb{N},\mathbb{R}^+)$ such that* $\inf f(j)>0$. *If* $\inf|\Lambda(\ln(f))|>0$, *then every solution of* $x_{n+1}=f(n)x_n$ *is AP.*

PROOF: Since $\inf|\Lambda(\ln(f))|>0$, then from Proposition 3.2, we obtain $\sum_{j=0}^{n}\ln(f(j))\in\mathcal{AP}(\mathbb{N},\mathbb{R})$. This implies $\ln(\prod_{j=0}^{n}f(j))\in\mathcal{AP}(\mathbb{N},\mathbb{R})$, and consequently $\prod_{j=0}^{n}f(j)$ is AP. □

7 The equation $x_{n+1}=f(n)x_n+g(n)$

Consider the difference equation

$$x_{n+1}=f(n)x_n+g(n), \tag{7.1}$$

where $f(n)$ is a nonconstant element of $\mathcal{AP}(\mathbb{N},\mathbb{R})$ and $g(n)$ is a nonzero element of $\mathcal{AP}(\mathbb{N},\mathbb{R})$. Observe that from Proposition 6.2, nonexistence in the homogeneous case implies uniqueness for Eq. (7.1). Now, it remains whether a unique AP solution exists.

Proposition 7.1. *If $M_{|f|}(0) < 1$, then every solution of Eq. (7.1) is bounded.*

PROOF: Any solution of Eq. (7.1) is given by

$$x_n = f_{n-1,0}x_0 + f_{n-1,1}g(0) + f_{n-1,2}g(1) + ... + g(n-1) = f_{n-1,0}x_0 + \sum_{j=0}^{n-1} f_{n-1,j+1}g(j).$$

Thus

$$|x_n| \leq \max\{|x_0|, ||g||_\infty\} \sum_{j=0}^{n-1} |f_{n-1,j}|.$$

The rest of the proof follows from part (iii) in Lemma 6.1. □

Theorem 7.1. *If $M_f^* > 1$, then Eq. (7.1) has a unique AP solution on $\mathbb{N}$.*

PROOF: As the uniqueness is obvious, we proceed with the existence. Let $M_f^* = B > 1$. Any solution of Eq. (7.1) can be written explicitly as

$$x_n = f_{n-1,0}\left(x_0 + \sum_{k=0}^{n-1}(f_{k,0})^{-1}g(k)\right).$$

Next, we fix $x_0 = -\sum_{k=0}^{\infty}(f_{k,0})^{-1}g(k)$ and show it is well-defined. Since

$$|x_0| \leq \sum_{k=0}^{\infty}|f_{k,0}|^{-1}|g(k)| < ||g||_\infty \sum_{k=0}^{\infty}|f_{k,0}|^{-1},$$

then by Lemma 6.1, the right hand side of the inequality is bounded. Next, we show that x_0 provides the required unique AP solution. Observe that

$$x_n = -\sum_{j=n}^{\infty} g(j)\prod_{i=n}^{j}(f(k))^{-1} = -\sum_{j=n}^{\infty} g(j)(f_{j,n})^{-1}$$

which implies $|x_n| \leq ||g||_\infty \sum_{j=n}^{\infty}|f_{j,n}|^{-1}$. Thus, it is bounded by part (iii) of Lemma 6.1. It remains to show the almost periodicity. Subtract the two equations

$$x_{n+p+1} = f(n+p)x_{n+p} + g(n+p) \quad \text{and} \quad x_{n+1} = f(n)x_n + g(n)$$

to obtain

$$|f(n+p)|\,||x_{n+p}-x_n| \le |x_{n+p+1}-x_{n+1}| + |x_n|\,|f(n+p)-f(n)| + |g(n+p)-g(n)|;$$

consequently,

$$||x_{n+p}-x_n||_\infty \le C_1||f(n+p)-f(n)||_\infty + C_2||g(n+p)-g(n)||_\infty,$$

where $C_1 = ||x_n||_\infty/(||f||_\infty - 1)$ and $C_2 = 1/(||f||_\infty - 1)$. Observe that $M_f^* > 1$ implies $||f||_\infty > 1$. Now, it is clear than an $\frac{\epsilon}{2C_1}$-translation number of f and an $\frac{\epsilon}{2C_2}$-translation number of g imply an ϵ-translation number of x_n. □

If $M_f^* < 1$ and $f, g \in \mathcal{AP}(\mathbb{Z},\mathbb{R})$, then it is well known that $x_0 = \sum_{k=-\infty}^{-1} f_{0,k+1}g(k)$ provides the unique AP Solution of equation (7.1) (Halanay and Rasvan [24]). However, for $f, g \in \mathcal{AP}(\mathbb{N},\mathbb{R})$, the above choice of x_0 is not well-defined, thus requiring a different approach.

Theorem 7.2. *In Eq. (7.1), each of the following holds true.*

(i) *If* $\inf_n |f(n)| > 1$, *then there exists a unique AP solution.*

(ii) *If* $||f||_\infty < 1$, *then there exists a unique AP solution which is globally asymptotically stable.*

PROOF: Part (i) follows from Theorem 7.1; however, we find it interesting to give a different proof using the Banach fixed point theorem. Define $M := ||f||_\infty$, $m = \inf_n |f|$ and $\alpha := \frac{1}{2}(M+2)$. So $m > 1$ and $\alpha > 1$. Now, let us consider the equation

$$y_{n+1} = \alpha y_n + (f(n)-\alpha)x_n + g(n).$$

If x_n is an AP sequence, then $g^*(n) = (f(n)-\alpha)x_n + g(n)$ is an AP sequence; hence, our equation becomes $y_{n+1} = \alpha y_n + g^*(n)$. By Theorem 4.1, this equation has a unique AP solution y_n which depends mainly on the choice of the AP sequence x_n. Define the operator $T: \mathcal{AP}(\mathbb{N}) \to \mathcal{AP}(\mathbb{N})$ such that $T(X) = Y$, where $X = \{x_n\} \in \mathcal{AP}(\mathbb{N})$ and $U = \{u_n\}$ is the unique AP solution of

$$u_{n+1} = \alpha u_n + (f(n)-\alpha)x_n + g(n).$$

Now, subtract $T(X) = U$ from $T(Y) = V$ to obtain

$$||T(Y) - T(X)||_\infty = ||V - U||_\infty.$$

From Eq. 4.4), we obtain

$$||U - V||_\infty < \frac{||f(n)-\alpha||_\infty}{\alpha - 1}||Y - X||_\infty.$$

Observe that $m \leq f(n) \leq M$ implies

$$\frac{m-\alpha}{\alpha-1} \leq \frac{f(n)-\alpha}{\alpha-1} \leq \frac{M-\alpha}{\alpha-1}.$$

Since $m > 1$, then $m - \alpha > 1 - \alpha = (\alpha - 1)(-1)$ and $\frac{m-\alpha}{\alpha-1} > -1$. Also, the choice of α implies $\frac{M-\alpha}{\alpha-1} < 1$. Thus, T is a contraction and by Banach fixed point theorem, T has a fixed point which is the unique AP solution of Eq. (7.1).

To prove (ii), we define $\alpha := \frac{1}{3}(1 - M)$. As in (i),

$$||T(Y) - T(X)||_\infty = ||V - U||_\infty$$

and

$$v_{n+1} - u_{n+1} = \alpha(v_n - u_n) + (f(n) - \alpha)(y_n - x_n).$$

However, this implies

$$||T(Y) - T(X)||_\infty \leq \frac{||f(n) - \alpha||_\infty}{1-\alpha} ||Y - X||_\infty.$$

Observe that $-M < f(n) < M$ implies

$$-1 < \frac{(-2M-1)}{(2+M)} < \frac{f(n)-\alpha}{1-\alpha} < \frac{4M-1}{2+M} < 1.$$

Therefore, T has a fixed point which is the unique AP solution of Eq. (7.1). Finally, the global asymptotic stability is trivial. □

Proposition 7.2. *An AP solution of Eq. (7.1) satisfies* $\mathrm{Mod}(x_n) \subseteq \mathrm{Mod}(f(n)) \cap \mathrm{Mod}(g(n))$.

PROOF: Consider Eq. (7.1) to obtain

$$||x_{n+p} - x_n||_\infty \leq C_1||f(n+p) - f(n)||_\infty + C_2||g(n+p) - g(n)||_\infty,$$

where $C_1 = ||x_n||_\infty/(||f||_\infty - 1)$ and $C_2 = 1/(||f||_\infty - 1)$. Now, $T(f, \frac{\epsilon}{2C_1})$ and $T(g, \frac{\epsilon}{2C_2})$ are contained in $T(x_n, \epsilon)$, from this fact and Proposition 3.4 the result follows. □

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