Extension, embedding and global stability in two dimensional monotone maps

Ahmad Al-Salman†, Ziyad AlSharawi*‡ and Sadok Kallel‡

† Department of Mathematics
Sultan Qaboos University, P. O. Box 36
PC 123, Al-Khod, Sultanate of Oman

‡ Department of Mathematics and Statistics
American University of Sharjah, P. O. Box 26666
University City, Sharjah, UAE

July 18, 2019

Abstract

We consider the general second order difference equation $x_{n+1} = F(x_n, x_{n-1})$ in which F is continuous and of mixed monotonicity in its arguments. In equations with negative terms, a persistent set can be a proper subset of the positive orthant, which motivates studying global stability with respect to compact invariant domains. In this paper, we assume that F has a semi-convex compact invariance, then make an extension of F on a rectangular domain that contains the invariance. The extension preserves the continuity and monotonicity of F. Then we use the embedding technique to embed the dynamical system generated by the extended map into a higher dimensional dynamical system, which we use to characterize the asymptotic dynamics of the original system. Some illustrative examples are given at the end.

AMS Subject Classification: 39A10, 39A30.

Keywords: Compact invariance, embedding, monotone maps, global stability.

1 Introduction

Second order difference equations $x_{n+1} = F(x_n, x_{n-1})$ in which F is non-decreasing in one argument and non-increasing in the other are known as second order difference equations of mixed monotonicity. Such equations have been investigated in the literature [5, 8–10, 16]

*Corresponding author: zsharawi@aus.edu

when the positive orthant is invariant (forward invariant), and the particular interest is usually given to the stability of equilibrium solutions (the fixed points of the map F). This interest stems from the fact that the dynamics of the equilibria can be sufficient to characterize the dynamics of the system. For instance, when $f(x)$ is continuous, decreasing, positive, $f(0) > 1$ and $xf(x)$ is increasing, the positive quadrant is an invariant of the equation $x_{n+1} = F(x_n, x_{n-1}) = x_n f(x_{n-1}) + h,\ h > 0$, and the unique positive equilibrium is a global attractor [2,15]. Under various levels of assumptions on the map F, several efforts have been successful in furnishing results of paramount significance in characterizing the asymptotic behaviour [7,9,12,13,17,18]. The work of Kulenovic et al. [12,13] assumes the system $(x, y) = (F(x, y), F(y, x))$ has no nontrivial solution, and relies heavily on the assumption that the map F assumes an invariant box to prove global stability of the unique equilibrium. A more general approach was given by Gouze and Hadeler [7] and developed further by Smith [17,18]. This approach relies on the notion of embedding the dynamical system generated by F into a larger symmetric monotone dynamical system (the references cited in [17,18] give a good account of the development of this approach), then the dynamics of the embedded system can be used to characterize the dynamics of the original system.

In discrete systems with negative coefficients, the positive orthant may not serve as an obvious invariant domain [1]. Furthermore, when an invariance is found, it may not be a box, which makes the developed theory in the aforementioned literature not enough to address the asymptotic dynamics. In this paper, we are concerned with continuous two dimensional maps of mixed monotonicity that assume a compact invariant domain, which is not necessarily a box. For writing and reading conveniences, we use $F(\uparrow, \cdot)$ or $F(\downarrow, \cdot)$ to denote that F is non-decreasing or non-increasing in its first variable, respectively. Similarly, for the second variable. Thus, our interest in this paper is limited to maps that satisfy $F(\uparrow,\uparrow)$, $F(\uparrow,\downarrow)$, $F(\downarrow,\uparrow)$ or $F(\downarrow,\downarrow)$. We focus on $F(\uparrow,\downarrow)$, and prove that for a continuous map $z = F(x, y)$ with mixed monotonicity on a compact and connected invariant domain Ω, if $T(x, y) = (F(x, y), x) : \Omega \to \Omega$ and F can be extended to $\widetilde{F}$ over a rectangular invariant domain containing Ω such that the system $(\widetilde{F}(x, y), \widetilde{F}(y, x)) = (x, y)$ has no solutions other than the fixed points of F, then $x_{n+1} = F(x_n, x_{n-1})$ must converge to a fixed point of F for all $(x_0, x_{-1}) \in \Omega$. We achieve this task based on several results. First, we give a method for constructing an extension in section two. Then in section three, we embed the extended system into a higher dimensional system using several auxiliary maps, then discuss the characteristics of the embedded system and its effect on the asymptotic behavior of the original systems. In section four, we consider a concrete case and apply the developed theory to establish global stability with respect to a certain invariant domain. Finally, we end this paper by some discussion and conclusion.

2 Invariant domains and map extension

This paper considers the second order difference equation

$$x_{n+1} = F(x_n, x_{n-1}), \quad \text{where} \quad n = 0, 1, \ldots \quad \text{and} \quad x_{-1}, x_0 \geq 0. \tag{2.1}$$

where F is continuous on a compact and connected invariant domain $\Omega \subset [0, \infty) \times [0, \infty) =: \mathbb{R}^{+2}$. Here Ω is invariant in the sense that if $(x, y) \in \Omega$, then $(F(x, y), x) \in \Omega$. Obviously, F must have at least one fixed point x^*, i.e., $F(x^*, x^*) = x^*$, which is called an equilibrium solution (or a steady state solution) of Eq. (2.1). Furthermore, we may consider $F : \Omega \to [m, M]$, where

$$m := \min_{(x,y)\in\Omega} F \quad \text{and} \quad M := \max_{(x,y)\in\Omega} F,$$

and by focusing on the second iterate of Eq. (2.1), there is no loss of generality if we assume $\Omega \subseteq [m, M] \times [m, M] =: [m, M]^2$.

Throughout this paper, we consider two partial order relations on Ω, namely, the southeast order $\leq_{se}$ and the northeast order $\leq_{ne}$. The southeast order is defined as

$$(x, y) \leq_{se} (u, v) \quad \text{if and only if} \quad x \leq u \quad \text{and} \quad v \leq y,$$

while the northeast order is defined as

$$(x, y) \leq_{ne} (u, v) \quad \text{if and only if} \quad x \leq u \quad \text{and} \quad y \leq v.$$

A well known observation [7, 17, 18] is that if a function $F : I^2 \to I$, with $I = [a, b]$ or $I = \mathbb{R}^+$, satisfies $F(\uparrow, \downarrow)$, then the symmetric map

$$G(x, y) = (F(x, y), F(y, x)) \tag{2.2}$$

is non-decreasing with respect to $\leq_{se}$, i.e., $G(\uparrow)$. This simple observation has had useful applications in the literature [7, 17, 18] for it allows to deduce global convergence properties for F from the dynamics of the symmetric map G in favorable situations (see §3). In general however, given a map $F : \Omega \to [m, M]$, Ω may not necessarily be a rectangular region, and the map G is not immediately well-defined. (see the examples of §4). This phenomenon makes the above definition of the symmetric map G ill-defined. In this section, we go around this problem by showing that we can extend a continuous map $F : \Omega \to [a, b]$, $\Omega \subset [a, b]^2$, into a map $\widetilde{F} : [a, b]^2 \to [a, b]$ that satisfies the following conditions:

(C1) $\widetilde{F}$ is continuous on $[a, b]^2 \supseteq \Omega$.

(C2) $F(x, y) = \widetilde{F}(x, y)$ for all $(x, y) \in \Omega$.

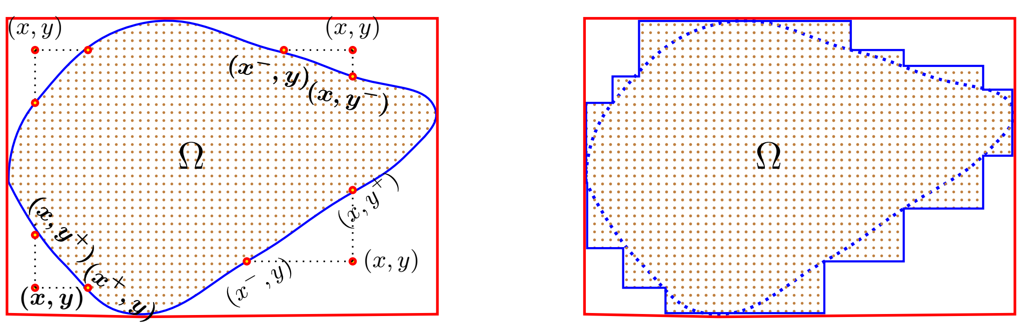
(C3) $\widetilde{F}$ has the same monotonicity behaviour as F. In particular, $\widetilde{F}(\uparrow,\downarrow)$ whenever $F(\uparrow,\downarrow)$, and $\widetilde{F}(\downarrow,\uparrow)$ whenever $F(\downarrow,\uparrow)$.

Throughout this paper, whenever talking about an extension $\widetilde{F}$, $\widetilde{F}$ must satisfy C1, C2 and C3. It is interesting to point out that the image set of our extension $\widetilde{F}$ is the same as that of F. We formalize these concepts in the following definition:

Definition 2.1. *Let F be a continuous function of mixed monotonicity on a domain $\Omega \subset \mathbb{R}^2$. A continuous function $\widetilde{F}$ is called* an extension *of F if the domain of F is contained in the domain of $\widetilde{F}$ and $\widetilde{F}$ is also of same mixed monotonicity. Further, $\widetilde{F}$ is called* a nice extension *if the values of $\widetilde{F}$ coincide with the values of F.*

2.1 Extending F from a convex domain to a rectangular domain

Assume Ω is compact, and we start by assuming the boundary $\partial\Omega$ to be a piecewise smooth Jordan curve (simple closed). We find it convenient to focus on the extension of $F(\downarrow,\uparrow)$. The extension of $G(\uparrow,\downarrow)$ can be obtained by considering $\widetilde{G}(x,y) = \widetilde{F}(y,x)$. For a point $p = (x,y)$ in the plane, we write (x^+,y) (respectively (x^-,y)) the point obtained as the intersection of the horizontal line through p with the curve $\partial\Omega$ on the right (respectively on the left), if this point exists. Similarly we write (x,y^+) (respectively (x,y^-)) the point obtained as the intersection of the vertical line through p with the curve $\partial\Omega$ on the top (respectively on the bottom). Part (a) of Fig. 2.1 illustrates our notations and the projection of an external point to the boundary $\partial\Omega$.



(a) (b)

Figure 2.1: In part (a) of this figure, we illustrate our notation of projecting a point (x,y) to the boundary $\partial\Omega$. In part (b), we illustrate the notion of putting the invariant domain Ω inside an Origami domain.

The next result gives a condition and a property on F which are needed in the sequel.

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^2$ be compact such that the boundary $\partial\Omega$ forms a Jordan curve. Suppose that F is continuous on Ω. If $\mathbf{r} : I \longrightarrow \partial\Omega$ is a parametrization so that $f := F \circ \mathbf{r}$ is in $\mathcal{C}^1([0,1])$*

and f' has only finitely many zeroes, then $\partial\Omega$ can be decomposed into finite number of pieces for which f is monotonic.

Proof. Let $0 \leq t_1 < t_2 < ... < t_m \leq 1$ be the zeroes of f'. Then f' has a single sign in each interval $(t_i, t_{i+1}), 1 \leq i \leq m-1$. Otherwise, if f' changes sign in (t_i, t_{i+1}) for some i, then by Dini's theorem, f' must have a zero in (t_i, t_{i+1}) which is impossible. Since $\partial\Omega$ forms a Jordan curve, there exists a continuous map $\mathbf{r} : [0,1] \to \mathbb{R}^2$ such that $\mathbf{r}(0) = \mathbf{r}(1)$ and the restriction of $\mathbf{r}$ to $[0,1)$ is injective. From the assumption that $f \in \mathcal{C}^1([0,1])$ and f' has only finitely many zeroes, for each $t \in [0,1]$, there exists an open interval $I_t \subseteq [0,1]$ such that $f'(t) \geq 0$ or $f'(t) < 0$. Now, $\cup_t I_t$ forms an open cover of $[0,1]$. The compactness implies the existence of a finite subcover, say $I_0, I_1, \ldots, I_n$. Hence, f is monotonic on each piece of the finite subcover, which completes the proof. □

Consider a portion of $\partial\Omega$ parametrized by $\mathbf{r}(t) = (x(t), y(t), I \subset [0,1]$, for which $F \circ \mathbf{r} : I \to \mathbb{R}$ is monotonic. We explain how to extend F on a rectangular domain containing that portion.

Lemma 2.2 (Ray Extension)**.** *Let F be continuous and of mixed monotonicity on a compact and convex set $\Omega \subset \mathbb{R}^2$. Suppose that $\mathbf{r}(t)$, $t_0 \leq t \leq t_1$ is a counterclockwise parametrization of a portion of $\partial\Omega$ on which $f = F \circ \mathbf{r}$ is monotonic. Then F can be extended to a function of same mixed monotonicity on the portion of a rectangular domain bounded on the left by $\mathbf{r}$.*

Proof. Again our proof is given for $F(\downarrow, \uparrow)$. If $\mathbf{r}(t)$ is of the form $(x(t), c)$ (horizontal) or $(c, y(t))$ (vertical) on $[t_0, t_1]$, then nothing needs to be done. Otherwise we proceed in cases. If $f(\uparrow)$, we have three cases to tackle as shown in Part (a) of Fig. 2.2. Part (b) of Fig. 2.2 covers the case $f(\downarrow)$. For a point $p = (x, y)$ out of the shaded region, the arrow in each case indicates the direction of the projection we take. For instance, if p is in the unshaded region of (iv) in Part (a), then

$$\widetilde{F}(x,y) = \begin{cases} F(x,y), & \text{if } (x,y) \in \Omega \\ F(x^-, y), & \text{if } (x,y) \in \Omega^c, \end{cases}$$

and if p is in the unshaded region of (ii) in Part (a), then

$$\widetilde{F}(x,y) = \begin{cases} F(x,y), & \text{if } (x,y) \in \Omega \\ F(x, y^-), & \text{if } (x,y) \in \Omega^c. \end{cases}$$

The other cases follow the same principle. □

Extensions as in Lemma 2.2 above are dubbed "ray extensions" since $\widetilde{F}$ is always constant on either horizontal or vertical rays of Ω^c. Ray extensions allow us in turn to extend our map F from convex Ω to a polygonal domain whose edges are horizontal or vertical, and each edge

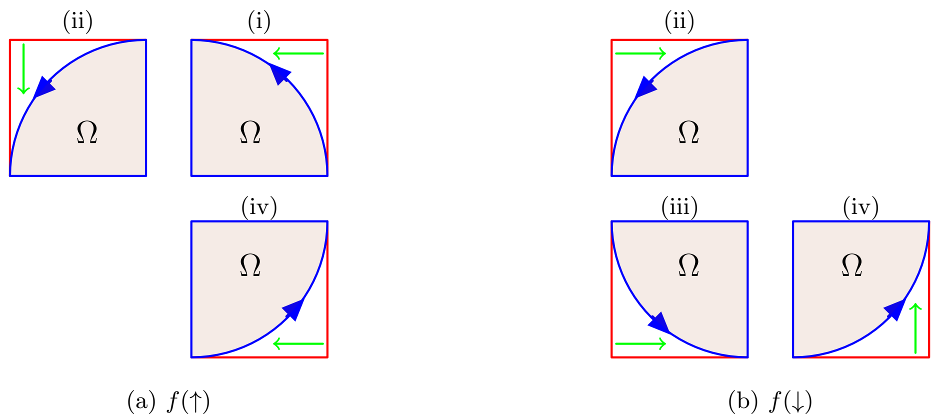


Figure 2.2: This figure shows the possible options for the boundary of a convex Ω and a possible ray extension. Part (a) is based on the assumption that f is non-decreasing and Part (b) is based on the assumption that f is non-increasing. Note that the missing quarter in Part (a) is due to the fact that we cannot have $f(\uparrow)$ and $F(\downarrow,\uparrow)$ at the same time. Similarly for the missing quarter of Part (b).

must intersect $\partial\Omega$. We call such a domain an "origami domain". Part (b) of Fig. 2.1 illustrates such a domain containing Ω. From now on we will have to assume that the boundary of Ω is parameterized as a piecewise regular curve[1] $\mathbf{r} : [0,1] \longrightarrow \mathbb{R}^2$ that is counterclockwise oriented and such that the derivative of $f = F \circ \mathbf{r}$ has finitely many zeros, so that by Lemma 2.1, F restricted to Ω is monotonic on a finite number of pieces.

Lemma 2.3. *Let Ω be a compact and convex region in the plane of whose boundary is a piecewise smooth Jordan curve, and let F be a continuous mixed monotonic function on Ω as in Lemma 2.1. Then F extends to a function $\widetilde{F}$ on an origami domain. Furthermore, this extension can be chosen so that its image coincides with that of F.*

Proof. Our curve $\partial\Omega$ is oriented counterclockwise bounding a convex region Ω. It has a parameterization $\mathbf{r} : [0,1] \to \partial\Omega$, $\mathbf{r}(0) = \mathbf{r}(1)$. For a connected portion C_j of the curve C, we will write its restricted parameterization as $\mathbf{r}_j : I_j \subset [0,1] \to C_j$, where $\mathbf{r}_j = \mathbf{r}_{|I_j}$.

By assumption (see Lemma 2.1), the curve splits into a finite number of segments $C_1, C_2, \ldots, C_n$ such that (i) these segments are either entirely horizontal or vertical, or of the six forms given in Figure 2.2, and (ii) the restrictions $f_j := F \circ \mathbf{r}_j$ are monotonic (increasing or decreasing) from $I_j = [a_j, b_j] \subset [0,1]$ into $\mathbb{R}$.

For simplicity, we say that we "fill in" a region if we are able to extend F over that region. If C_j is entirely vertical or horizontal, then nothing needs to be done. Otherwise,we extend F to the right of the curve (here "right" is with respect to the orientation which is counterclockwise). The extension fills in the corresponding box depicted in Figure 2.2. The region to be filled has horizontal top and bottom sides, left boundary C_j (again "left" w.r.t the counterclockwise orientation) and right most boundary the line segment having vertex

[1]A *regular* curve is a curve whose derivative at every point exists and is non-zero.

$(\mathbf{r}_j(a_j)_x, \mathbf{r}_j(b_j)_y)$ or $(\mathbf{r}_j(a_j)_y, \mathbf{r}_j(b_j)_x)$ depending on the situation (we use the notation that if $p \in \mathbb{R}^2$, then p has coordinates (p_x, p_y)). We now extend F for the six cases by taking the appropriate ray extension as described in Lemma 2.2 and depicted in Figure 2.2. Going around our boundary curve $\partial\Omega$, with Ω convex, and extending F to $\widetilde{F}$ on those boxes to the right of the C_j's, we obtain an extension of F to a domain that is an origami domain. $\square$

The next step is to extend from the origami domain into a rectangular domain. To that end, we will define the notion of "grafting". Let $\gamma_i : I_i \to \mathbb{R}^3$, $i = 1, 2$, be two curves in space. The following process explains how to "graft" the first curve γ_1 to a point of γ_2. We write $p_0 = \gamma_1(0)$ the starting point of γ_1. For a point $p = \gamma_2(t)$ on the second curve, we consider the new curve

$$\gamma_p : I_1 \to \mathbb{R}^3 \quad , \quad \gamma_p(s) = \gamma_1(s) + \overrightarrow{p_0 p}$$

This is the curve $\mathbf{r}_1$ translated along the vector $\overrightarrow{p_0 p}$ so it starts at the point $\gamma_2(t)$; i.e. $\gamma_p(0) = \gamma_2(t)$. We say that "we graft γ_1 along γ_2" means that we graft γ_1 at every point of γ_2, obtaining a new replica of γ_1 starting at each point of γ_2. For suitable choices of γ_1 and γ_2, this process yields a surface in space as illustrated in Figure 2.3. The operation of grafting is not commutative.

The following example is easy to check and is what we use in the sequel.

Example 2.1. *Let γ_1 and γ_2 be defined by $\gamma_1(s) = (x_0, s, z(s))$ and $\gamma_2(t) = (t, y_0, w(t))$, with $s, t \in [0, 1]$. Grafting γ_1 along γ_2 yields the parameterized surface $\nu : [0, 1]^2 \to \mathbb{R}^3$ defined by*

$$\nu(t, s) = (t, y_0 + s, z(s) + w(t) - z(0)).$$

In particular, if z and w are monotonous, then ν is mixed monotonous. If the surface is seen as a graph of F in the variables $(x = t, y = y_0 + s)$, then $F(\uparrow, \downarrow)$ if $z \uparrow$ and $w \downarrow$.

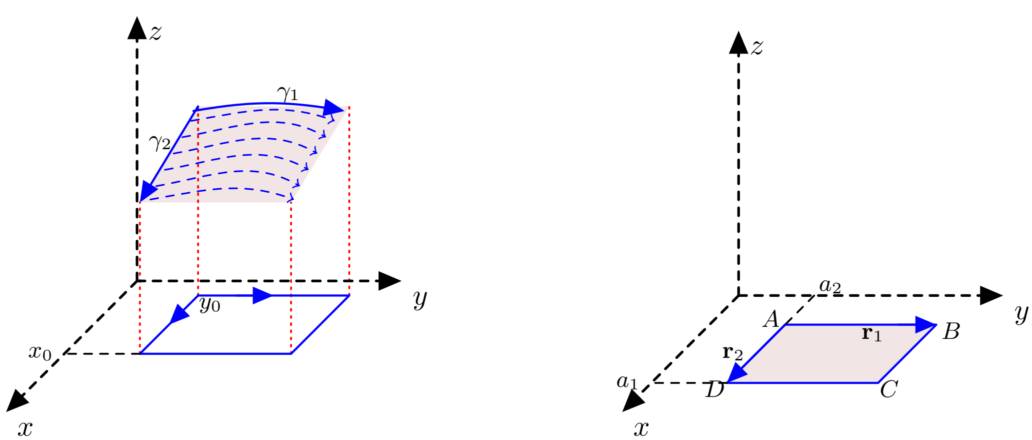


Figure 2.3: Grafting γ_1 along γ_2

The above example suggests how to extend a function of two variables F over a rectangular planar region, knowing F on one vertical and one horizontal side. Let A, B, C, D be the vertices of the rectangular region, $A = (a_1, a_2)$, and parameterize AB by $\mathbf{r}_1(s) = (a_1, \alpha(s))$ for $s \in I_1$, and parameterize AD by $\mathbf{r}_2(t) = (\beta(t), a_2)$ for $t \in I_2$. We then obtain curves in space

$$\gamma_1(s) = (a_1, \alpha(s), F(a_1, \alpha(s))) \quad , \quad s \in I_1 \tag{2.3}$$

(this is the graph of F over AB), and

$$\gamma_2(t) = (\beta(t), a_2, F(\beta(t), a_2)) \quad , \quad t \in I_2.$$

We now graft γ_1 along γ_2 to get the surface

$$\nu(s,t) = (\beta(t), \alpha(s), F(a_1, \alpha(s)) + F(\beta(t), a_2) - F(a_1, a_2)).$$

For a point $(x, y) = (\beta(t), \alpha(s))$ in the interior of the rectangular region, we can set

$$\widetilde{F}(x,y) = F(a_1, y) + F(x, a_2) - F(a_1, a_2)$$

Evidently if $F(\downarrow, \uparrow)$ on $AB \cup AD$, meaning $\alpha \uparrow$ and $\beta \downarrow$, then as well $\widetilde{F}(\downarrow, \uparrow)$.

Corollary 2.1 (Filling by Grafting)**.** *If $F(\downarrow, \uparrow)$ on $AB \cup AD$, then $\widetilde{F}$ is a well-defined extension on the interior of $ABCD$ with $\widetilde{F}(\downarrow, \uparrow)$.*

Remark 2.1. *(*Reverse Grafting*) We can graft a curve γ_1 at a point $\gamma_2(t)$ by translating γ_1 so that now the end point of γ_1 ends at $\gamma_2(t)$. We will call this process "reverse grafting". Recall that normal grafting is about translating γ_1 so its starting point is that $\gamma_2(t)$. This is used as follows. Suppose now that our rectangle $ABCD$ is parameterized counterclockwise, γ_1 is the graph of F over BA (with parameterization from B to A), and γ_2 the graph over AD. As in Corollary 2.1, a filling of the rectangle can be done by grafting γ_2 along γ_1, or by reverse grafting γ_1 along γ_2. We skip the details as this is now straightforward.*

We are now in a position to prove our first extension result.

Theorem 2.1 (Convex Case)**.** *Let Ω be a compact and convex region in the plane of whose boundary is a piecewise smooth Jordan curve, circumscribed on a rectangle R. Then any continuous function F of mixed monotonicity on Ω with hypothesis as in Lemma 2.1 extends to a function $\widetilde{F}$ on R. Furthermore, this extension can be chosen so that its image coincides with that of F.*

Proof. Based on Lemma 2.3, we start from the point where Ω is an origami domain. In this case, we need to consider four cases as shown in Fig. 2.4 and extend F over the unshaded rectangles. Each region labeled (I) through (IV) has a horizontal edge and a vertical edge.

Each edge is a portion of the parameterized boundary, positively oriented according to the arrow. We label γ_{top} (resp. γ_{bot}) the curve obtained as the graph of F restricted to the top edge (resp. bottom edge), and γ_{ver} the graph of F restricted to the vertical edge. The extension $\widetilde{F}(\downarrow,\uparrow)$ is obtained by performing the following filling:

- Region (I): grafting γ_{ver} along γ_{top} (or reverse grafting γ_{top} along γ_{ver})
- Region (II): grafting γ_{top} along γ_{ver}
- Region (III): grafting γ_{bot} along γ_{ver}
- Region (IV): grafting γ_{ver} along γ_{bot}.

In both cases (I) and (IV), the intermediate value theorem shows that grafting doesn't change the range. This is not the case for the extensions in cases (II) and (III). If we insist on not getting out of the range of F, we need to use an alternative extension. Fortunately this is possible in those two cases. For the region (II) we set

$$\widetilde{F}(x,y) = \begin{cases} F(x,y), & (x,y) \in \Omega \\ \min\left(F(x^-,y), F(x,y^+)\right), & (x,y) \in \Omega^c. \end{cases} \tag{2.4}$$

It is easy to check that $\widetilde{F}(\downarrow,\uparrow)$. Similarly for region (III), we use the extension

$$\widetilde{F}(x,y) = \begin{cases} F(x,y), & (x,y) \in \Omega \\ \max\left(F(x,y^-), F(x^+,y)\right), & (x,y) \in \Omega^c. \end{cases} \tag{2.5}$$

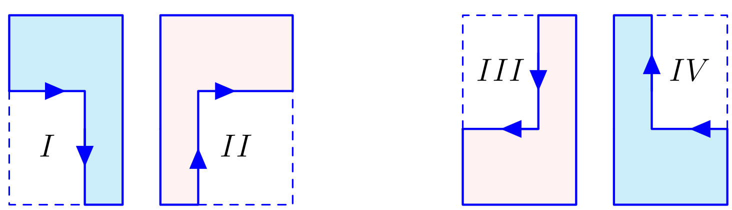


Figure 2.4: The various cases for the boundary pieces of an origami domain.

These are well-defined extensions and their range stays within the range of f. This concludes the proof and the existence of an extension for convex domains. □

2.2 Extending F from a semi-convex domain to a rectangular domain

We believe that Theorem 2.1 is true for all domains Ω, parameterized by $\mathbf{r}$, so that the derivative of $f = F \circ \mathbf{r}$ has finitely many zeroes on its domain. This means that the extension to a rectangular region containing Ω is possible on such domains. We do not verify this claim to this fullest extent, but we restrict ourselves to generalizing the theorem to semi-convex domains. This covers a much wider family of domains Ω other than the convex ones.

A planar region Ω with boundary $C = \partial\Omega$ a simple curve, is called semi-convex if any ray emanating from $p \in C$ into Ω^c, which is either horizontal or vertical doesn't intersect the curve again. Part (a) of Fig. 2.5 (a) illustrates one such domain.

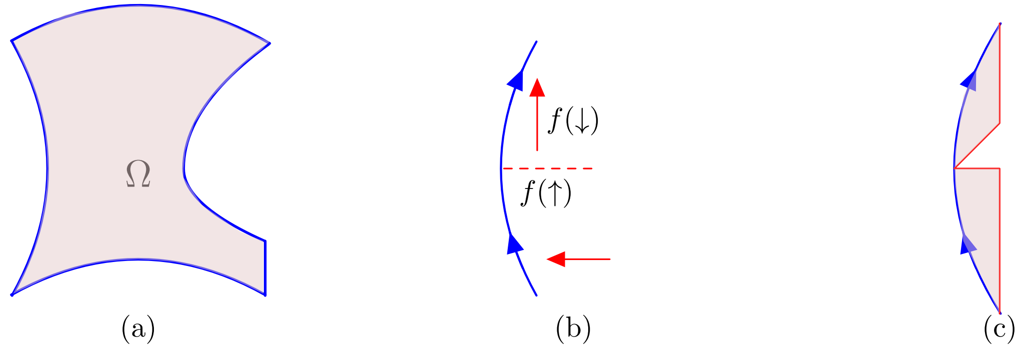


Figure 2.5: A semi-convex domain is shown in Part (a). Part (b) shows the right boundary of Ω parametrized and oriented counterclockwise. $f(t) = F(r(t))$ changes monotonicity on this side of the boundary. Part (c) shows the filling technique we follow.

Theorem 2.2 (Semi-Convex Case)**.** *Suppose Ω is semi-convex and $F(\downarrow, \uparrow)$ with hypothesis as in Lemma 2.1. Then F extends to $\widetilde{F}(\downarrow, \uparrow)$ on a larger rectangular domain containing Ω. Furthermore, the range of $\widetilde{F}$ coincides with the range of F.*

Proof. Recall that we are writing our proofs based on mixed monotonicity of the form $F(\downarrow, \uparrow)$. By hypothesis $f := F \circ \mathbf{r}$, where $\mathbf{r}$ is a parametrization of C, is monotonic on a finite number of segments $C_1, \ldots, C_n$, $C = C_1 \cup \cdots \cup C_n$ (see Lemma 2.1). We can proceed as in the convex case, extending F first on an origami domain. In fact, this is all that needs to be checked. To get to the origami domain, we extend F on a box containing C_i for all i by performing either a horizontal or a vertical ray extension. Here however we might run into the following problem illustrated by Part (b) of Fig. 2.5.

Indeed, suppose C_1, C_2 are consecutive segments of C, $f_1 = F \circ \mathbf{r}_1$ is $\uparrow$ while $f_2 = F \circ \mathbf{r}_2$ is $\downarrow$. We can extend to the indicated boxes as in Lemma 2.2 by performing the ray extension shown by the projection arrows. When this is done, we obtain a discontinuity along the ray emanating from $p = C_1 \cap C_2$. Along that ray, $\widetilde{F}_1$ and $\widetilde{F}_2$ the corresponding extensions for C_1 and C_2 do not necessarily agree. To go around this problem, we extend as before but leave a sector as shown in Part (c) of Fig. 2.5. This extension is well-defined and continuous, but now we have to fill in the sector. A sector has two parameterized edges: the first edge E_1 and the second edge E_2, where first and second are with respect to increasing t in the parametrization. For each sector in Fig. 2.6, there are cases to consider describing the various combinations of $f_1 \uparrow\downarrow$ on E_1 and $f_2 \uparrow\downarrow$ on E_2. We treat in details the case of (d), the other three being similar.

Suppose we have a sector as in Part (d) of Fig. 2.6. Note that the function f_1 is always increasing $\uparrow$ on E_1. If f_2 is decreasing, then we can proceed as follows. Let (x, y) be a point in this sector, (x, y^-) its vertical projection in E_1 and (x, y^+) its vertical projection in E_2. We can parameterize the segment $[(x, y^-), (x, y^+)]$ by an arc-length so that a point there is of the form

$$(x, y) = \left(1 - \frac{y - y^-}{y^+ - y^-}\right)(x, y^-) + \frac{y - y^-}{y^+ - y^-}(x, y^+)$$

and consequently we can define a linear extension of F as follows

$$\widetilde{F}(x, y) = \left(1 - \frac{y - y^-}{y^+ - y^-}\right) F(x, y^-) + \frac{y - y^-}{y^+ - y^-} F(x, y^+).$$

Such an extension is continuous by construction, and we can verify that $\widetilde{F}(\downarrow, \uparrow)$; further, the range of $\widetilde{F}$ is within the range of F.

Suppose now that $f_2 \uparrow$ (still dealing with Figure 2.6 (d)). The linear extension above doesn't yield in general the same mixed monotonicity extension. Instead and in this case, we can graft γ_2 along γ_1 where as is now understood γ_i is the graph of F over parameterized E_i (see Eq. (2.3) and Fig. 2.3). In this case as well, the range of $\widetilde{F}$ stays within the range of F.

The other extensions in cases (a),(b) and (c) work the same. Note that for (a), f_1 must be decreasing and f_2 is increasing, because of the nature of $F(\downarrow, \uparrow)$. Here only linear extension is required. Same for (c).

Finally, by carrying out these extensions over sectors we end up extending F over a domain which is again an Origami domain, meaning all edges are either vertical or horizontal. By trying to extend this new F to a rectangular domain, as we've done for the convex case, we run into the novel situation where we have to fill a region surrounded by three sides as in Part (a) of Fig. 2.7. This is done as indicated in that figure: we first extend partially by ray extension, then fill-in the resulting sector. We skip writing the details as they are self-evident.

□

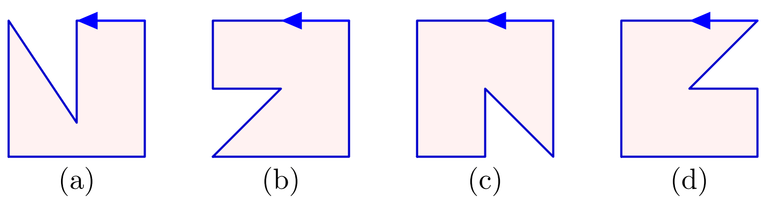
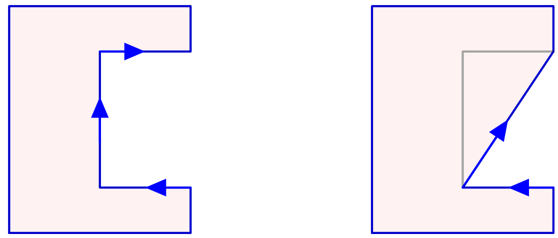


Figure 2.6: Extending over a semi-convex curve reduces to filling-in the sectors labeled (a) through (d)

3 Posets and the embedding technique

The embedding technique can be used to study the asymptotic behaviour of the orbits in Eq. (2.1) whenever F is of mixed monotonicity and assumes an invariant box. In its simplest form, this approach can be summarized as follows [7, 17].



(a) (b)

Figure 2.7: (a) Filling a rectangle where F is known on three sides. (b) Extending by ray extension then filling the resulting sector.

Theorem 3.1. *Suppose $F : \Omega := [a,b]^2 \to [a,b]$ and $F(\uparrow,\downarrow)$. Then the system*

$$x_{n+1} = F(x_n, x_{n-1}), \quad \textit{where} \quad n = 0, 1, \ldots \quad \textit{and} \quad x_{-1}, x_0 \geq 0,$$

has a global attractor provided the map $G : \Omega \to \Omega$, $G(x,y) = (F(x,y), F(y,x))$, has a unique fixed point (x^*, x^*).

The rest of the section elaborates on this result and points to alternative embeddings (i.e. other choices of maps G that can play a similar role as in Theorem 3.1). The fixed points of the embedded map G play a key role in Theorem 3.1, and they are the solutions of the simultaneous equations $F(x,y) = x$ and $F(y,x) = y$. Obviously, the fixed points of F are solutions. We call a fixed point of G that is not a fixed point of F *an artificial fixed point* of F.

A Riesz space is a vector space endowed with a partial order $\leq$ which makes it a lattice and satisfies translation invariance and positive homogeneity [14]. For example, $\mathbb{R}^n$ with $\leq_{se}$, $\leq_{ne}$, $\leq_{sw}$ or $\leq_{nw}$ ordering is a Riesz space. The north-east partial ordering corresponds to the componentwise order (i.e. $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$), which is the most recognizable Riesz partial ordering on $\mathbb{R}^n$. A Riesz space is Archimedean if it also satisfies the following condition: for every pair of positive elements x and y, there exists an integer n such that $nx \geq y$. Order convergence and topological convergence coincide on Archimedean Riesz spaces, and in fact much more is true for finite dimensional Riesz space. The following can be found in [14].

Theorem 3.2. *Let $(E, \leq)$ be a Riesz space of finite dimension $n \geq 1$. Then E is Archimedean if and only if it is isomorphic to $\mathbb{R}^n$ under its componentwise order.*

This means in particular that the analog of the "monotone sequence theorem" applies in the Archimedean case.

Corollary 3.1. *In a finite dimensional Archimedean Riesz space, a bounded increasing chain $x_1 \leq x_2 \leq \cdots$ converges to its least upper bound. The convergence is topological.*

A function $f : X \to X$ is order preserving with respect to the partial ordering of X if whenever $x \leq y$, $f(x) \leq f(y)$. The following is an immediate consequence of Corollary 3.1.

Proposition 3.1. *Let X be a closed subspace of an Archimedean Riesz space $(E, \leq)$ with a minimal element 0 and maximal element 1 for $(X, \leq)$. Let $f : X \to X$ be any continuous order preserving function. Then there are fixed points a^* and b^* of f, so that for any $x \in X$, we have*

$$a^* \leq \liminf f^n(x)\ ,\ \limsup f^n(x) \leq b^*.$$

Proof. Since 0 is minimal, $0 \leq f(0)$, and so using the order preserving properties of f, $\{f^n(0)\}$ is an increasing chain in X. It must converge to its least upper bound, say a^* in X. Since f is continuous and the convergence is topological, this implies that a^* must be a fixed point of f. Similarly, $\{f^n(1)\}$ must be decreasing converging to a fixed points, say b^* in X. Finally and for any $x \in X$, $0 \leq x \leq 1$ so that $f^n(0) \leq f^n(x) \leq f^n(1)$ for all $n \geq 1$. From this the claim follows. Notice that when $a^* = b^*$, $\{f^n(x)\}$ converges to a fixed point of f, independently of $x \in X$. □

Corollary 3.2. *[7, 17] Let $F : \Omega := [a, b]^2 \to [a, b]$ be a continuous function that satisfies $F(\uparrow, \downarrow)$, and let $G : \Omega \to \Omega$ be the symmetric map $G(x, y)) = (F(x, y), F(y, x))$. For any initial condition $X = (x, y) \in \Omega$, the sequence $\{G^n(X)\}$ is either convergent to a fixed point $X^* = (x^*, y^*)$ of G or*

$$(x^*, y^*) \leq_{se}\ \liminf G^n(x, y), \limsup G^n(x, y) \leq_{se}\ (y^*, x^*).$$

Proof. As indicated already, G is order preserving for the $\leq_{se}$ order. This is easy to see: if $(x, y) \leq_{se} (u, v)$, then the two facts

$$F(x, y) \leq F(u, y) \leq F(u, v) \qquad \text{and} \qquad F(y, x) \geq F(v, x) \geq F(v, u)$$

imply $G(x, y) = (F(x, y), F(y, x)) \leq_{se} (F(u, v), F(v, u)) = G(u, v)$. Set $X = [a, b]^2$, $f = G$, the minimal element $0 = (a, b)$, the maximal element $1 = (b, a)$. We have $a^* = lub\{G^n(a, b)\}$ and $b^* = glb\{G^n(b, a)\}$. Clearly if $a^* = (x^*, y^*)$, then $b^* = (y^*, x^*)$. □

Notice that the diagonal of Ω, i.e., $\{(x, x) : a \leq x \leq b\}$ is invariant under G. When G doesn't have a unique fixed point, and thus has a pair of symmetric fixed points, convergence established under G is not generally helpful in establishing convergence under F since forward obits of F, $\mathcal{O}_F^+(x_0, x_{-1}) = \{x_{-1}, x_0, x_1, \ldots\}$, where $x_{n+1} = F(x_n, x_{n-1})$, can be scrambled under G. The following example illustrates such an unfavorable situation.

Example 3.1. Let $F(x, y) = xf(y)$, where f is bounded and decreasing with $f(0) > 1$, then the Jacobian matrix of $G(x,y) = (F(x,y), F(y,x))$ is given by

$$J\left(\begin{bmatrix} F(x,y) \\ F(y,x) \end{bmatrix}\right) = \begin{bmatrix} f(y) & xf'(y) \\ yf'(x) & f(x) \end{bmatrix}.$$

Thus, at the positive equilibrium x^* of F, we obtain

$$J(x^*, x^*) = \begin{bmatrix} 1 & x^* f'(x^*) \\ y^* f'(x^*) & 1 \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix at (x^*, x^*) are

$$\lambda_1 = 1 - x^* f'(x^*) > 1 \quad \text{and} \quad \lambda_2 = 1 + x^* f'(x^*) < 1.$$

Hence, (x^*, x^*) cannot be stable, and therefore, it cannot globally attract Ω under the iteration of G. On the other hand, when $-2 \leq x^* f'(x^*) < 0$, (x^*, x^*) is a saddle, and consequently, a one dimensional manifold of Ω is attracted to (x^*, x^*). Indeed, it is easy to show in this case that the diagonal of Ω is attracted under G to (x^*, x^*). In conclusion, if $F(x,y) = xf(y)$, then (x^*, x^*) cannot attract all of Ω in Corollary 3.2, and therefore, there must be at least two other asymmetric fixed points of G.

Example 3.1 above prompts us to seek alternative choices of a symmetric map G that makes a clearer connection between the orbits of F and the orbits of the symmetric system. We provide two options below (Corollaries 3.3 and 3.4). A third option for defining G has been considered in [17].

Corollary 3.3. *Suppose that $F : \Omega \to [a,b]$ is continuous and satisfies $F(\uparrow, \downarrow)$. Consider the map $G : \Omega \times \Omega \to \Omega \times \Omega$ defined by*

$$G((x,y),(u,v)) = ((F(x,y), u), (F(u,v), x)). \tag{3.1}$$

Then, for each initial conditions $X, Y \in \Omega$, the omega limit set $\omega((X,Y), G)$ is either a fixed point (X^*, Y^*) of G or belongs to an interval of the form $I := [(X^*, Y^*), (Y^*, X^*)]_{se}$ in which the endpoints are fixed points of G.

Proof. Consider $\Omega = [a,b] \times [a,b]$ and $\Omega \times \Omega$ with the southeast order $\leq_{se}$. Define the map

$$g : \Omega \times \Omega \to \Omega \quad \text{as} \quad g((x,y),(u,v)) = (F(x,y), u).$$

It can be checked that $g(\uparrow, \downarrow)$, and by writing

$$G((x,y),(u,v)) = (g((x,y),(u,v)), g((u,v),(x,y)))$$

we can see that G is order preserving with respect to $\leq_{se}$. Set $A := (a,b)$ and $B := (b,a)$. If

$X = \Omega \times \Omega$, then $(X, \leq_{se})$ is a closed subspace of an Archimedean Riesz space with minimal element (A, B) and maximal element (B, A). The conclusion follows from Proposition 3.1. □

Since the map G as defined in Eq. (3.1) is of particular interest to us, the next proposition summarizes some of its dynamics.

Proposition 3.2. *Let $X = (x, y)$ and $Y = (u, v) \in \Omega$. Each of the following holds true:*

(i) If $x \leq v \leq y \leq u, u \leq F(u, v)$ and $F(x, y) \leq x$, then

$$G(X, Y) \leq_{se} (X, Y) \leq_{se} (X, X) \leq_{se} (Y, X) \leq_{se} G(Y, X).$$

In particular, the same inequalities hold true if $(x, y) = (v, u)$ and $F(x, y) \leq x \leq y \leq F(y, x)$.

(ii) If $v \leq x \leq u \leq y, F(u, v) \leq u$ and $x \leq F(x, y)$, then

$$(X, Y) \leq_{se} G(X, Y) \leq_{se} G(X, X) \leq_{se} G(Y, X) \leq_{se} (Y, X).$$

In particular, the same inequalities hold true if $(x, y) = (v, u)$ and $x \leq F(x, y), F(y, x) \leq y$.

Proof. For $X = (x, y)$ and $Y = (u, v)$, $G(X, Y) \leq_{se} (X, Y)$ is equivalent to $(F(x, y), u) \leq_{se} X$ and $Y \leq_{se} (F(u, v), x)$. Therefore,

$$G(X, Y) \leq_{se} (X, Y) \qquad \Leftrightarrow \qquad \begin{cases} F(x, y) \leq x, \\ u \leq F(u, v), \\ y \leq u, \text{ and } x \leq v \end{cases}$$

and

$$(X, Y) \leq_{se} G(X, Y) \qquad \Leftrightarrow \qquad \begin{cases} x \leq F(x, y), \\ F(u, v) \leq u, \\ v \leq x \text{ and } u \leq y. \end{cases}$$

All claims follow immediately. □

Corollary 3.4. *Let $F : \Omega \to [a, b]$ be a continuous function such that $F(\uparrow, \downarrow)$, and consider the symmetric map $G : \Omega^2 \times \Omega^2 \to \Omega^2 \times \Omega^2$ defined by*

$$G((X, Y), (U, V)) = (((F(X), F(V)), X), ((F(U), F(Y)), U)).$$

Then for each initial condition $(X, Y), (U, V) \in \Omega^2$, the sequence $\{G^n((X, Y), (U, V))\}$ either converges to a fixed point $((X^*, X^*), (U^*, U^*))$ of G, or lies in an interval between two such fixed points.

Proof. The proof follows the same lines once it is shown that $G(\uparrow)$ with respect to $\leq_{se}$ on $\Omega^4 = \Omega^2 \times \Omega^2$. This can be done as follows: give Ω and Ω^4 the southeast ordering $\leq_{se}$, and give Ω^2 the northeast ordering $\leq_{ne}$. Define the maps

$$g_1 : \Omega \times \Omega \to \Omega \quad \text{as} \quad g_1((x,y),(u,v)) = (F(x,y), F(u,v)).$$

and

$$g : \Omega^2 \times \Omega^2 \to \Omega^2 \quad \text{as} \quad g((X,Y),(U,V)) = (g_1(X,V), X).$$

We claim that $g_1(\uparrow,\downarrow)$ and $g(\uparrow,\downarrow)$. Consider $X' = (x', y')$ and let $(X,Y),(X',Y) \in \Omega \times \Omega$ such that $X \leq_{se} X'$. Then

$$g_1(X,Y) = (F(X), F(Y)) \text{ and } g_1(X',Y) = (F(X'), F(Y)).$$

Since $x \leq x'$ and $y' \leq y$, we obtain

$$F(X) = F(x,y) \leq F(x',y) \leq F(x',y') = F(X'),$$

which implies $g_1(X,Y) \leq_{se} g_1(X',Y)$. Next, consider $Y' = (u',v')$, and let $(X,Y),(X,Y') \in \Omega \times \Omega$ such that $Y \leq_{se} Y'$, then $u \leq u', v \geq v'$. Thus,

$$F(Y) = F(u,v) \leq F(u',v) \leq F(u',v') = F(Y'),$$

which implies $g_1(X,Y') \leq_{se} g_1(X,Y)$. Similarly one shows that $g(\uparrow,\downarrow)$, from which we deduce that $G(\uparrow)$. Finally it is straightforward to find that the fixed points of G are obtained from the solutions of $(F(x,y), F(y,x)) = (x,y)$. The rest of the claim follows from Proposition 3.1. □

4 Examples

To apply our developed theory on maps of mixed monotonicity, a compact invariance has to be found, then an extension map $\widetilde{F}$ on a rectangular domain has to be constructed. If $\widetilde{F}$ has no artificial fixed points, i.e., the solution of $(\widetilde{F}(x,y), \widetilde{F}(y,x)) = (x,y)$ gives the same fixed points as those of F, then every orbit of F converges to a fixed point of F. To achieve all of this, examples can be involved and lengthy; however, since our purpose here is to give illustrative examples of the developed theory in the previous sections, we give relatively short examples. In particular, we give two examples. The first has an invariant rectangular domain in which the known theory can be used to prove global stability. The second lacks a known invariant rectangular domain, but a compact invariant domain is found. In this case, our our developed approach is used.

4.1 First example

Consider the difference equation

$$y_{n+1} = F(y_n, y_{n-1}) = \frac{p + qy_n}{1 + y_n + ry_{n-1}}, \quad \text{where } 0 < p \leq q \text{ and } r > 0 \tag{4.1}$$

and the initial conditions are non-negatives. This problem has been discussed in the literature in which partial stability results have been obtained for specific choices of the parameters [3–5, 10, 11]. We are not interested here in writing a survey about the obtained results in the literature, but rather, we are interested in illustrating how to apply the extension-embedding approach here. Note that F is increasing in its first argument and decreasing in its second one within the range of our parameters. Furthermore, F has a unique positive fixed point $\bar{y}$, which is less than q. In fact, this positive fixed point is locally asymptotically stable for all values of the parameters. Now, we show the existence of an invariant compact domain. Since

$$0 < F(x, y) \leq \frac{q(1 + x)}{1 + x + ry} < q,$$

then the orbits of Eq. (4.1) enter the rectangular region $\mathcal{S} := [0, q] \times [0, q]$ and stay. Thus, $\mathcal{S}$ serves as a compact invariant rectangular domain of $T(x, y) = (F(x, y), x)$. In this case, no need for an extension of F since the embedding theory of Smith [17, 18] or the global stability results of Kulenovic and Merino [13] can be applied. Next, we investigate the solution of the system

$$F(x, y) = x, \quad F(y, x) = y \quad \text{and} \quad (x, y) \in \mathcal{S}. \tag{4.2}$$

Obviously, $x = y = \bar{y}$ is a solution, which is a fixed point of F. We need to exclude the existence of artificial fixed points.

Proposition 4.1. *Consider the parameters p, q and r as given in Eq. (4.1). If one of the following cases is satisfied, then the system in (4.2) has no artificial fixed points in $\mathcal{S}$.*

(i) $q \leq 1$ *(ii)* $0 \leq r \leq 1$ *(iii)* $r > 1$ *and* $p > \frac{1}{4}(r-1)(q-1)^2$.

Proof. (i) Since $y - x = F(y, x) - F(x, y)$ implies $x + y = q - 1$, solutions (x, y) have to be located on the line $x + y = q - 1$, which does not go through the positive quadrant. To verify the second case, we eliminate y from $F(x, y) = x$ and $F(y, x) = y$ to obtain

$$\left[(r-1)x^2 - (r-1)(q-1)x + p\right]\left[(-r-1)x^2 + (q-1)x + p\right] = 0.$$

The factor on the right gives the fixed point $x = y = \bar{y}$. The factor on the left gives the artificial fixed points. Thus, we focus on the solutions of $(r-1)x^2 - (r-1)(q-1)x + p = 0$. When $r = 1$, we obtain no solution. When $0 < r < 1$, we obtain a negative solution and a positive one. Thus, the artificial fixed points are in the second and fourth quadrants. To verify Part (iii), observe the restrictions give non-real roots. □

Now, we apply Theorem 3.1 to obtain the following global stability result.

Corollary 4.1. *Consider Eq. (4.1) and assume that one of the restrictions in Proposition 4.1 is satisfied, then the positive equilibrium* $(\bar{y}, \bar{y})$ *is globally asymptotically stable with respect to the positive quadrant.*

4.2 Second example

Consider the function $F(x, y) = \frac{p+qx}{1+x+ry} - h, \quad q \geq p > h > 0$ and $r > 0$, i.e., we consider the difference equation

$$x_{n+1} = F(x_n, x_{n-1}) = \frac{p + qx_n}{1 + x_n + rx_{n-1}} - h, \quad \text{where } q \geq p > h > 0 \tag{4.3}$$

and $n = 0, 1, \ldots, x_{-1}, x_0 \geq 0$. This equation has a rich dynamics to the extent that well-defined solutions are not obvious anymore. Since our objective here is to give an illustrative example, we limit our attention to $r = 1$ and $q = 2p$. In this case, we have a unique positive equilibrium given by $x^* = b - h$. The next result characterizes local stability.

Lemma 4.1. *Consider Eq. (4.3) with* $r = 1, q = 2p$ *and* $0 < h < \min\left\{p, \frac{p+1}{2}\right\}$. *The unique positive equilibrium of Eq. (4.3) is locally asymptotically stable.*

Proof. The Jacobian matrix of F at $x^* = b - h$ has trace $T = \frac{p}{1+2x^*}$ and determinant $D = T$. Thus, $h < p$ makes x^* positive and $D < 1$ makes it locally asymptotically stable. □

When $h = \frac{1}{2}$, F has an infinite number of artificial fixed points, which makes this case a special case that needs different type of treatment. Therefore, we proceed with the assumption $0 < h < \frac{1}{2}$. In the next result, we establish a compact invariant domain.

Lemma 4.2. *Consider Eq. (4.3) with* $q = 2p, r = 1$ *and* $0 < h < \min\{p, \frac{1}{2}\}$. *Let* $c = \frac{x^*}{h}(x^* + p + 1)$. *The compact region bounded by the octagon of vertices* $A_0 : (c, c)$, $A_1 : (x^*, c)$, $A_2 : (0, x^*)$, $A_3 : (0, 0)$ *and* $A_4 : (c, 0)$ *forms an invariant domain.*

Proof. Let Ω be the compact region bounded by the boundary curves $\gamma_j(t) = (1 - t)A_j + tA_{j+1 \bmod 5}$, $j = 0, \ldots, 4$ and $0 \leq t \leq 1$. We show that Ω is invariant under $T(x, y) = (F(x, y), x)$. Since T is one-to-one on the positive quadrant, it is sufficient to show that $T(\gamma_j) \subset \Omega$. Observe that $x^* < \frac{x^*}{h} < c$. The rest of the proof is computational, and we give the main outlines here. We have $T(\gamma_0) = T(c + t(x^* - c), c)$ with $T(\gamma_0(0)) = A_1$ and $T(\gamma_0(1)) = A_2$. Since also the curve is convex, we obtain $T(\gamma_0) \subset \Omega$. $T(\gamma_1) = T((1-t)x^*, (1-t)c + tx^*)$, which is increasing in x, decreasing in y, and satisfies $T(\gamma_1(0)) = A_2$, $T(\gamma_1(1)) = \left(\frac{p}{x^*+1} - h, 0\right)$. Thus, $T(\gamma_1) \subset \Omega$. $T(\gamma_2) = \left(\frac{p}{1+(1-t)x^*} - h, 0\right) \subset \gamma_3$.

$$T(\gamma_3) = T(tc, 0) = \left(\frac{p(1 + 2ct)}{1 + ct} - h, ct\right).$$

The second component increases from 0 to c, while the first component increases from x^* to $x^* + \frac{pc}{1+c}$. It remains to show that $x^* + \frac{pc}{1+c} < c$. We have

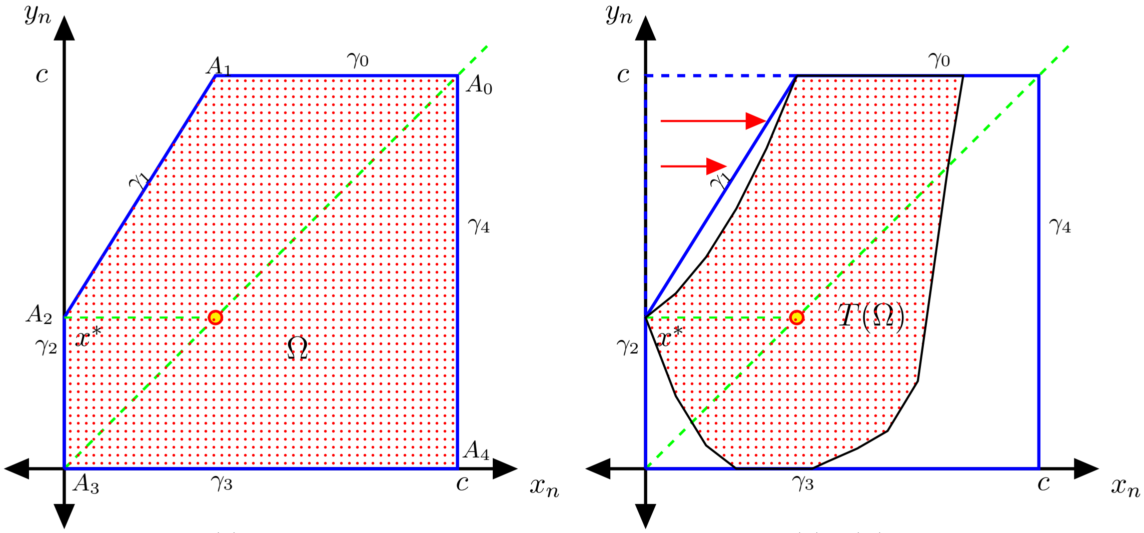
$$(c - x^*)(1+c) - pc > 0 \quad \Leftrightarrow \quad \frac{x^*}{h^2}\left(4p^2(p-3h+1) + (1-3h)^2 p + h^2(1-2h)\right) > 0,$$

which holds true when $0 < h < \{p, \frac{1}{2}\}$. Finally $T(\gamma_4) \subset \gamma_0$. Hence, the proof is complete (Fig. 4.1 gives an illustration). $\square$

Next, we consider the invariant region Ω as the domain of F, and construct a nice extension $\widetilde{F}$ on the rectangular region $\mathcal{S} = [0,c]^2$. Since $F(\gamma_1(t))$ is increasing when $0 < t < 1$ and $0 < h < \frac{1}{2}$, our nice extension will be

$$\widetilde{F}(x,y) = \begin{cases} F(x,y), & \text{if } (x,y) \in \Omega \\ F(x^+,y), & \text{if } (x,y) \in \mathcal{S} \setminus \Omega, \end{cases}$$

where $x^+ = \frac{(y-x^*)x^*}{(c-x^*)}$. Next, we show that $\widetilde{F}$ has no artificial fixed points.



(a) Ω is shaded.

(b) $T(\Omega)$ is shaded.

Figure 4.1: Part (a) of this figure shows the invariant region Ω, while part (b) shows $T(\Omega)$ together with the extension through horizontal projections. The scale is missing to indicate the general form of the region when $0 < h < \min\{p, \frac{1}{2}\}$.

Proposition 4.2. *The map $\widetilde{F}$ has no artificial fixed points in $\mathcal{S} = [0,c]^2$.*

Proof. For $(x,y) \in \Omega$, $\widetilde{F}(x,y) = F(x,y)$, and it is straightforward to find that the only solution of $(F(x,y), F(y,x)) = (x,y)$ is (x^*, x^*) as long as $0 < h < \frac{1}{2}$. Next, consider $(x,y) \in \mathcal{S} \setminus \Omega$. This means we have to seek solutions (x,y) that satisfy

$$\widetilde{F}(x,y) = F(x^+,y) = F\left(\frac{(y-x^*)x^*}{c-x^*}, y\right) = x \quad \text{and} \quad F(y,x) = y, \tag{4.4}$$

where $y > \frac{c-x^*}{x^*}x + x^*$ and $0 < x < x^*$. Handling the two nonlinear equations together with the inequality is a cumbersome task, but we get around it by considering equations (4.4) together with $y = mx + x^*$, and show that no solution exist for $m > \frac{c-x^*}{x^*} = \frac{2x^*+1}{h}$. This covers the points in $\mathcal{S} \setminus \Omega$ except the y-axis. Thus, we investigate solutions of

$$F\left(\frac{(mx)x^*}{c-x^*}, mx + x^*\right) = x \qquad \text{and} \qquad F(mx + x^*, x) = mx + x^*.$$

For computational conveniences, substitute $p = x^* + h$ and eliminate x from the two equations to obtain a cubic equation in m of the form

$$a_3(x^*, h)m^3 + a_2(x^*, h)m^2 + a_1(x^*, h)m + a_0(x^*, h) = 0.$$

Replace m by $M + \frac{c-x^*}{x^*}$ to obtain

$$b_3(x^*, h)M^3 + b_2(x^*, h)M^2 + b_1(x^*, h)M + b_0(x^*, h) = 0. \tag{4.5}$$

The coefficients b_j have same sign, which can be verified by writing each b_j as a polynomial in x^*, then depend on the assumption that $0 < h < \frac{1}{2}$. We show b_3, which is

$$b_3(x^*, h) = h^3(1-h)^2(4(x^*)^3 + 4(x^*)^2 + (1-h)(3h+1)x^* + h(1-h-h^2).$$

The other coefficients are handled similarly. To this end, it has been shown that no $M > 0$ satisfies equation (4.5), and consequently no $m > \frac{c-x^*}{x^*}$. Finally, it remains to consider solutions of $\widetilde{F}(x, y) = x$ and $F(y, x) = y$ when $x = 0$, which is simple and has no possible solutions. Hence, the proof is complete. □

Now, we have all the needed machinery to prove global stability with respect to Ω.

Corollary 4.2. *Consider Eq. (4.3) with* $0 < h < \min\{p, \frac{1}{2}\}$. *The positive equilibrium is globally asymptotically stable with respect to the invariant set* Ω.

Proof. Proposition 4.2 shows that we have no artificial fixed points for $\widetilde{F}$ as long as $0 < h < \min\{p, \frac{1}{2}\}$. Now, Corollary 3.3 (or Corollary 3.4) gives the global stability for the extension $\widetilde{F}$ on $\mathcal{S}$, which gives the global stability for F on Ω. □

It is worth mentioning that it is possible to prove the global stability with respect to the persistent set rather than Ω, but it is a matter of computations to show that orbits of T get inside Ω in few iterations.

5 Conclusion and discussion

This paper is concerned with 2-dimensional continuous maps of mixed monotonicity that are defined on compact domains. When the compact domain is convex or semi-convex and the boundary is a piecewise smooth Jordan curve, a function F can be extended to a function $\widetilde{F}$ on a rectangular domain, having the same monotonicity properties, and the extension can be chosen so that its image coincides with that of F. In this case, we call it *nice extension.* We believe this result is generalizable to non-convex domains under proper conditions on the boundary; however, the general proof is technical. It can be developed further based on the need in real applications.

The embedding technique (Theorem 3.1) takes the following convenient form : Let Ω be a compact subset of $\mathbb{R}^2$ and $z = F(x, y)$ be continuous on Ω. Suppose that $F(\uparrow, \downarrow)$. If F has a nice extension
widetildeF over a rectangular domain containing Ω such that $\widetilde{F}$ has no artificial fixed points, then $x_{n+1} = F(x_n, x_{n-1})$ must converge to a fixed point of F for all $(x_0, x_{-1}) \in \Omega$. The concept of *artificial fixed points* is used to denote solutions of the system

$$(\widetilde{F}(x, y), \widetilde{F}(y, x)) = (x, y)$$

that are not fixed points of $\widetilde{F}$. The use of the this concept is driven by the fact that solutions of this system are in fact fixed points of the higher dimensional map G used in our embedding (see Corollary 3.3 or Corollary 3.4). This result can be considered as a high dimensional generalization of Coppel's result [6], which states the following: Suppose $f : [a, b] \to [a, b]$ is continuous. A necessary and sufficient condition that the iteration sequence $x_{n+1} = f(x_n)$ converge, whatever initial point $x_0 \in [a, b]$ is chosen, is that the equation $f(f(x)) = x$ has no roots except the roots of the equation $f(x) = x$.

Two examples have been considered. One of them shows the existence of an invariant box in which global stability has been obtained based on the available results in the literature without the need for an extension. The second example shows the existence of a compact invariant domain that is not a box. In this case, we constructed a nice extension and used our approach to obtain global stability with respect to the compact invariant domain.

Finally, it is worth mentioning that the main obstacle facing our approach is the solution of the system $(\widetilde{F}(x, y), \widetilde{F}(y, x)) = (x, y)$. Although the solution of the system can be unique with respect to the original map F, it may not be unique with respect to the extended map $\widetilde{F}$. From our perspective, this challenge can be tackled by seeking an alternative invariant domain, or by investigating the basin of attraction of each individual equilibrium point, i.e., the original equilibrium points as well as the artificial ones. However, this subject needs further research before a definite answer can be given.

References

[1] R. Abu-Saris, Z. AlSharawi, and M. B. Rhouma. The dynamics of some discrete models with delay under the effect of constant yield harvesting. *Chaos Solitons Fractals*, 54:26–38, 2013.

[2] Z. AlSharawi. A global attractor in some discrete contest competition models with delay under the effect of periodic stocking. *Abstr. Appl. Anal.*, pages 7, Art. ID 101649, 2013.

[3] A. M. Amleh, E. Camouzis, and G. Ladas. On second-order rational difference equation. I. *J. Difference Equ. Appl.*, 13(11):969–1004, 2007.

[4] A. M. Amleh, E. Camouzis, and G. Ladas. On the dynamics of a rational difference equation. II. *Int. J. Difference Equ.*, 3(2):195–225, 2008.

[5] E. Camouzis and G. Ladas. *Dynamics of third-order rational difference equations with open problems and conjectures*, volume 5 of *Advances in Discrete Mathematics and Applications.* Chapman & Hall/CRC, Boca Raton, FL, 2008.

[6] W. A. Coppel. The solution of equations by iteration. *Proc. Cambridge Philos. Soc.*, 51:41–43, 1955.

[7] J.-L. Gouzé and K. P. Hadeler. Monotone flows and order intervals. *Nonlinear World*, 1(1):23–34, 1994.

[8] E. A. Grove and G. Ladas. *Periodicities in nonlinear difference equations*, volume 4 of *Advances in Discrete Mathematics and Applications.* Chapman & Hall/CRC, Boca Raton, FL, 2005.

[9] V. L. Kocić and G. Ladas. *Global behavior of nonlinear difference equations of higher order with applications*, volume 256 of *Mathematics and its Applications.* Kluwer Academic Publishers Group, Dordrecht, 1993.

[10] M. R. S. Kulenović and G. Ladas. *Dynamics of second order rational difference equations.* Chapman & Hall/CRC, Boca Raton, FL, 2002. With open problems and conjectures.

[11] M. R. S. Kulenović, G. Ladas, L. F. Martins, and I. W. Rodrigues. The dynamics of $x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-1}}$: facts and conjectures. *Comput. Math. Appl.*, 45(6-9):1087–1099, 2003. Advances in difference equations, IV.

[12] M. R. S. Kulenović, G. Ladas, and W. S. Sizer. On the recursive sequence $x_{n+1} = (\alpha x_n + \beta x_{n-1})/(\gamma x_n + \delta x_{n-1})$. *Math. Sci. Res. Hot-Line*, 2(5):1–16, 1998.

[13] M. R. S. Kulenović and Orlando Merino. A global attractivity result for maps with invariant boxes. *Discrete Contin. Dyn. Syst. Ser. B*, 6(1):97–110, 2006.

[14] W. A. J. Luxemburg and A. C. Zaanen. *Riesz spaces. Vol. I.* North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1971. North-Holland Mathematical Library.

[15] G. Nyerges. A note on a generalization of Pielou's equation. *J. Difference Equ. Appl.*, 14(5):563–565, 2008.

[16] H. Sedaghat. *Nonlinear difference equations*, volume 15 of *Mathematical Modelling: Theory and Applications*. Kluwer Academic Publishers, Dordrecht, 2003. Theory with applications to social science models.

[17] H. L. Smith. The discrete dynamics of monotonically decomposable maps. *J. Math. Biol.*, 53(4):747–758, 2006.

[18] H. L. Smith. Global stability for mixed monotone systems. *J. Difference Equ. Appl.*, 14(10-11):1159–1164, 2008.