A global attractor in some discrete contest-population models with delay under the effect of periodic stocking*

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Abstract

In this paper, we consider discrete models of the form $x_{n+1} = x_n f(x_{n-1}) + h_n$, where h_n is a nonnegative p-periodic sequence representing stocking in the population, and investigate their dynamics. Under certain conditions on the recruitment function $f(x)$, we give a compact invariant region, and use Brouwer fixed point theorem to prove the existence of a p-periodic solution. Also, we prove the global attractivity of the p-periodic solution when $p = 2$. In particular, this study gives some theoretical results attesting to the belief that stocking (whether it is constant or periodic) preserves the global attractivity of the periodic solution in contest models with short delay. Finally, as an illustrative example, we discuss Peilou's model with periodic stocking.

Keywords: Discrete models; contest models; stocking; local stability; global attractor; Pielou's equation.
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1 Introduction

In mathematical ecology, difference equations of the form $x_{n+1} = x_n f(x_n), \quad n \in \mathbb{N} := \{0, 1, ...\}$ are used to model single-species with non-overlapping generations [9, 12], where x_n denotes the number of sexually mature individuals at discrete time n and $f(x_n)$ is the density-dependent net growth rate of the population. The form of the function $f(x)$ is chosen to reflect certain characteristics of the studied population such as intra-specific competition. For some background readings on the type of models obtained by the various choices of $f(x)$, we refer the reader to [6, 9, 29] in the discrete case and [10] and the references therein for the continuous case. Two classical types are known as the scrambled and contest competition models [29]. Our attention in this work is limited to contest competition models where $f(x)$ is assumed to be decreasing, $xf(x)$ is increasing and asymptotic to a certain level at high population densities. A prototype of such these models is the Beverton-Holt model [7], which is obtained by considering $f(x) = \frac{\mu K x}{K+(\mu-1)x}$. Here, $\mu > 1$ is interpreted as the growth rate per generation and K is the carrying capacity of the environment. In populations with substantial time needed to reach sexual maturity, certain delay effect must be included in the function $f(x)$, which motivates considering difference equations of the form

$$x_{n+1} = x_n f(x_{n-k}), \tag{1.1}$$

where k is a fixed positive integer [14]. In general, it is widely known that long time delay has a destabilizing effect on the population's steady state while short time delay can preserve stability [15, 17, 28]. However, when the delay is large, the dynamics of Eq. (1.1) is less tractable [11]; furthermore, we are more interested here on the effect of stocking than the effect of delay, and therefore, we keep the time delay short to preserve stability in the absence of stocking. In particular, we fix the delay to be $k = 1$.

A substantial body of research has explored the effect of constant stocking on population models without delay [16, 21–27]. In brief and general terms, it has been found that constant stocking can be used to suppress chaos, reverse the period doubling phenomena, lower the risk of extinction, and have a stabilizing effect on the population steady state. On the other hand, and to the best of our knowledge, little (if any) has been done to explore the effect of stocking (whether constant or periodic) on models with delay. So, our work here has a two-fold objective; to study the effect of periodic stocking on contest competition models with delay and to complement the work of the author and his collaborators in [5], where the dynamics of Eq. (1.1) with $k = 1$ was studied under the effect of constant yield harvesting. Recall that we have some accumulating restrictions on the function $f(x)$ due to the nature of the considered type of models. So, in an abstract mathematical form, our problem can be posed as follows: Consider the difference equation

$$x_{n+1} = x_n f(x_{n-1}) + h_n, \tag{1.2}$$

where $\{h_n\}$ is a non-negative p-periodic sequence representing stocking due to refuge, immigration, feeding, ...etc, and the function $f(x)$ obeys the following conditions:

(C1) $f(0) = b > 1$.

(C2) $f \in C^1([0, \infty))$ and $f(x)$ is decreasing on $[0, \infty)$.

(C3) $xf(x)$ is increasing and bounded.

The condition in (C1) is a generic one in the absence of stocking, i.e., if $b \leq 1$ and $h_n = 0$, then there is no long term survival regardless of the initial density of the population.

This paper is organized as follows: In Section two, we give some preliminary results concerning local stability, boundedness and global stability of Eq. (3.1) when the stocking sequence is 1-periodic, i.e., when $h_n = h > 0$ for all $n \in \mathbb{N}$. In Section three, the period of the stocking sequence is taken to be larger than one. A compact invariant region has been established and a characterization of the periodic solutions is given. Also, the global asymptotic behavior of solutions has been investigated when $p = 2$. As a particular case of Eq. (3.1), we discuss Pielou's equation with delay one in Section four.

2 Preliminary results: the autonomous case

In this section, we focus on the autonomous case, i.e., $h_j = h > 0$ for all $j = 0, 1, \ldots, p-1$. Thus, Eq. (1.2) becomes

$$x_{n+1} = x_n f(x_{n-1}) + h. \tag{2.1}$$

Some results concerning Eq. (2.1) can be found in the literature [18]; however, for the sake of completeness and usage in the nonautonomous case, we give the following preliminary results.

2.1 Local stability and boundedness

Eq. (1.2) has two equilibrium solutions, say $\bar{x}_{1,h}$ and $\bar{x}_{2,h}$. The smaller equilibrium $\bar{x}_{1,h}$ originates from the origin and slides downward as h increases, while the large equalibrium $\bar{x}_{2,h}$ originates from the positive equilibrium that exists in the absence of stocking and slides upward. Obviously, $\bar{x}_{1,h}$ is negative, and therefore beyond our interest. Since $\bar{x}_{2,h}$ is positive and increasing in h, $f(\bar{x}_{2,h}) < 1$ for all $h > 0$. The linearized equation associated with Eq. (2.1) at a fixed point $\bar{x}$ is given by

$$y_{n+1} - f(\bar{x})y_n - \bar{x}f'(\bar{x})y_{n-1} = 0. \tag{2.2}$$

Define $p := f(\bar{x})$ and $q := -\bar{x}f'(\bar{x})$. For $\bar{x} = \bar{x}_{2,h}$, we have $0 < p < 1$ and q is non-negative. The roots of $\lambda^2 - p\lambda + q = 0$ determine the local stability of our equilibrium point. Since $\lambda_{j,h} = \frac{1}{2}(p + (-1)^j\sqrt{p^2 - 4q})$, $j = 1, 2$, $\bar{x}_{2,h}$ starts as stable at $h = 0$ and stays stable as long as $q < 1$. Figure 1 clarifies the relationship between p, q and the magnitude of $\lambda_{j,h}$. We summarize these facts in the following proposition:

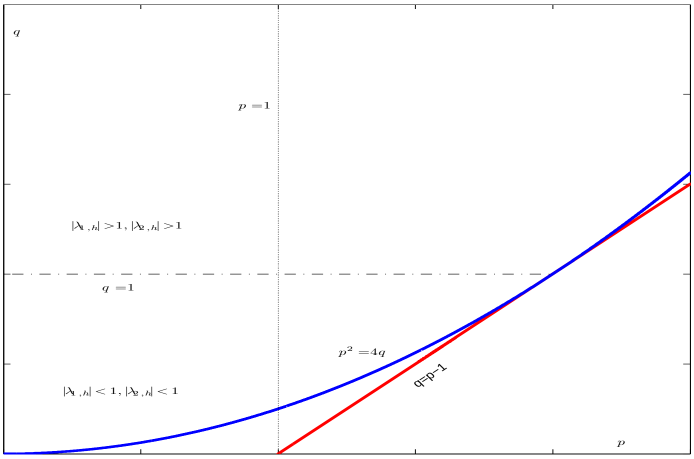

Figure 1: This figure shows the magnitude of the characteristic roots of Eq. (2.2) depending on the location of the (p, q) values, where $p = f(\bar{x})$ and $q = -\bar{x}f'(\bar{x})$.

Proposition 2.1. *Assume that conditions (C1) to (C3) are satisfied and $h > 0$. The positive equilibrium $\bar{x}_{2,h}$ of Eq. (2.1) is locally asymptotically stable.*

Proof. Since $\bar{x}_{2,h} > \bar{x}_{2,0}$, we have $p < 1$. Also, since $F(t) = tf(t)$ is increasing, we obtain $F'(t) = tf'(t) + f(t) > 0$ and consequently $-q + p > 0$. Thus, we have $q < p < 1$. Now, Figure 1 makes the rest of the proof clear. $\square$

It is obvious that $x_k \geq h$ for all $k \geq 1$. On the other hand, since

$$x_{n+1} = x_{n-1}f(x_{n-1})f(x_{n-2}) + hf(x_{n-1}) + h \leq x_{n-1}f(x_{n-1})b + hb + h,$$

the boundedness of $y = tf(t)$ assures the boundedness of all solutions of Eq. (1.2).

2.2 Oscillations and global stability

A solution of Eq. (2.1) is called oscillatory if it is neither eventually less than nor larger than $\bar{x}_{2,h}$ [11]. Also, one can consider oscillations about a curve [5]. A solution $\{x_n\}$ of Eq. (2.1) is called oscillatory about a curve $H(x,y) = 0$ if the sequence $\{u_n = (x_{n-1}, x_n)\}$ does not eventually stay on one side of the curve. The latter definition can be more convenient in some cases; however, in Eq. (2.1), both are equivalent when we consider $H(x,y) = y - x$ as we show in the following result:

Proposition 2.2. *A solution of Eq. (2.1) is oscillatory if and only if it is oscillatory about the curve $y = x$.*

Proof. Assume that $\{x_n\}$ oscillates about $\bar{x}_{2,h}$, but it is not oscillatory about $y = x$. So, $\{x_n\}$ is either eventually increasing or eventually decreasing, which contradicts the assumption that x_n is oscillatory about $\bar{x}_{2,h}$. Conversely, suppose $\{(x_{n-1}, x_n)\}$ oscillates about $y = x$, but $\{x_n\}$ does not oscillate about $\bar{x}_{2,h}$. First, we consider the case $x_n \leq \bar{x}_{2,h}$ for all $n \geq n_0$. If $x_m > x_{m-1}$ for some $m > n_0$, then $f(x_m) < f(x_{m-1})$ and consequently

$$x_{m+1} = x_m f(x_{m-1}) + h > x_m f(x_m) + h > x_m.$$

So we can induce an eventually increasing sequence which contradicts our assumption. If $x_m \leq x_{m-1}$ for some $m > n_0$, then $x_{m+1} \leq x_m f(x_m) + h$. Thus, either $x_{m+1} \leq x_m$ and the induction leads to a decreasing sequence that must converge which is not possible, or $x_{m+1} > x_m$ and we go back to the first scenario. Finally, the case $x_n \geq \bar{x}_{2,h}$ for all $n \geq n_0$ can be handled similarly. □

Next, we define the map

$$T(x,y) = (y, yf(x) + h). \tag{2.3}$$

The map T portrays the solutions of Eq. (2.1) geometrically in the nonnegative quadrant, and therefore, it plays a prominent role in the sequel. Here, we used the nonnegative quadrant to denote the positive quadrant union the axes on the boundary. By applying the map T on the regions above and below the curve $y = x$, one can observe that a non-equilibrium solution of Eq. (2.1) must be oscillatory. Also, using the map T, one can observe that stocking increases the frequency of oscillations in the following sense: The length of semi-cycles in the absence of stocking is longer than the length of semi-cycles in the existence of stocking, where a semi-cycle is used to denote the string of consecutive terms above or below the equilibrium.

Since solutions of Eq. (2.1) are bounded, we define

$$S := \limsup x_n \qquad \text{and} \qquad I := \liminf x_n. \tag{2.4}$$

From the equation $x_{n+2} = x_n f(x_n) f(x_{n-1}) + h f(x_n) + h$ and using the fact that $tf(t)$ is increasing, we obtain

$$S \leq Sf(S)f(I) + hf(I) + h \qquad \text{and} \qquad I \geq If(I)f(S) + hf(S) + h. \tag{2.5}$$

When $h > 0$, we have $S \geq I > 0$. So, we can multiply the first inequality by I and the second one by S to obtain

$$S(f(S) + 1) \leq I(f(I) + 1).$$

Since $t(f(t)+1)$ is increasing, we obtain $I = S$. This approach was used by Camouzis and Ladas in [8], and it was used by Nyerges in [18] to prove that $\bar{x}_{2,h}$ is globally attractive. This fact together with the local stability established in Proposition 2.1 shows the global asymptotic stability of $\bar{x}_{2,h}$ as we summarize in the following proposition:

Proposition 2.3. *The equilibrium solution $\bar{x}_{2,h}$ of Eq. (2.1) is globally asymptotically stable.*

Next, it is obvious that the positive quadrant forms an invariant for Eq. (2.1); however, since solutions are bounded, we are interested in a bounded invariant that can be developed to serve us in the periodic case. Notice that by invariance here we always mean forward invariance, i.e., R_h is an invariant of Eq. (2.1) if $T(x, y) \in R_h$ for all $(x, y) \in R_h$. To establish the existence of an invariance, we need to have in mind the following simple fact:

Proposition 2.4. *There exists a finite constant $c_h \geq h$ such that $G_h(t) = (bt + h)f(t) \leq bc_h$ for all $t \geq 0$. Furthermore, c_h can be taken as $c_h := \frac{1}{b} \sup_t G_h(t)$.*

Proof. Use the fact that $tf(t)$ is bounded and $f(t)$ is decreasing with $f(0) = b$ and $\lim_{t\to\infty} f(t) = 0$ to obtain the result. □

Next, define the curves $\Gamma_j, j = 0, 1, 2, 3, 4$ to be the line segments that connect the points $(0, 0), (0, h), (c_h, bc_h + h)$, $(bc_h + h, bc_h + h)$, $(bc_h + h, 0)$ and $(0, 0)$ respectively. Now, define R_h to be the region bounded by the curves of $\Gamma_j, j = 0, \ldots, 4$ including the boundary, then the following result gives a bounded invariant of Eq. (2.1). Here, it is worth mentioning that Γ_0 shrinks to a point at $h = 0$; however, our notation and arguments about the invariant region are still valid except the boundary of R_h becomes a quadrilateral rather than a pentagon.

Theorem 2.1. *The region R_h as defined above gives a compact invariant for Eq. (2.1).*

Proof. Consider the map $T(x, y)$ as defined in (2.3). T is one-to-one on the positive quadrant. Thus, all we need is to test T on the boundary of R_h. It is straightforward computations to find that $T(\Gamma_0) \subseteq \Gamma_1$. Since horizontal line segments are mapped to vertical line segments under T, we test the end points of Γ_2 to find

$$T(c_h, bc_h + h) = (bc_h + h, (bc_h + h)f(c_h) + h)$$

and

$$T(bc_h + h, bc_h + h) = (bc_h + h, (bc_h + h)f(bc_h + h) + h).$$

By the choice of c_h given in Proposition 2.4, we have

$$(bc_h + h)f(bc_h + h) + h \leq (bc_h + h)f(c_h) + h \leq bc_h + h.$$

Thus, $T(\Gamma_2) \subset \Gamma_3$. Next, $T(\Gamma_3) \subset R_h$ and $T(\Gamma_4) = (0, h)$ are straightforward to observe. Finally, we show that $T(\Gamma_1) \subset R_h$. For $0 \leq t \leq c_h$, we have

$$T(t, bt + h) = (bt + h, (bt + h)f(t) + h);$$

however, $h \leq bt + h \leq bc_h + h$ and $(bt + h)f(t) + h \leq bc_h + h$ by the choice of c_h, which completes the proof. Figure 2 illustrates the region R_h and its image under the map T when $(bt + h)f(t)$ is increasing. □

3 Periodic stocking

In this section, we force periodic stocking on Eq. (2.1) to obtain

$$x_{n+1} = x_n f(x_{n-1}) + h_n, \tag{3.1}$$

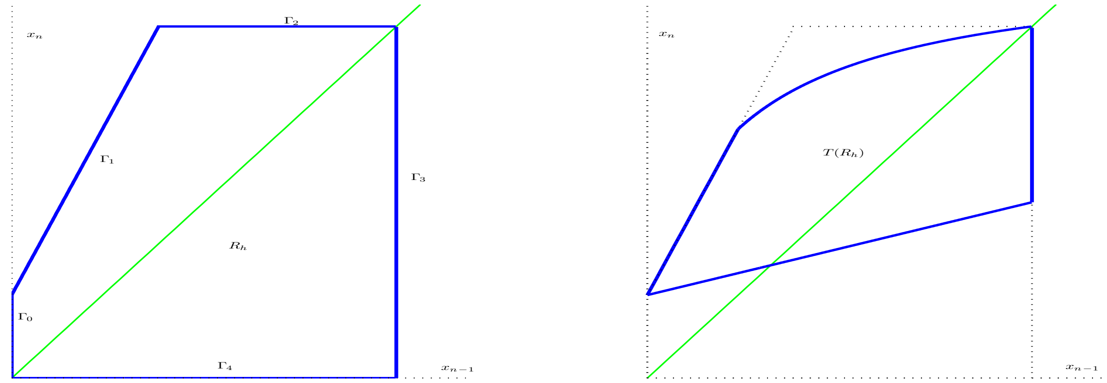

Figure 2: The figure on the left shows the choice of the compact region R_h when $y = (bt + h)f(t)$ is increasing, and the one on the right shows $T(R_h)$ with blue boundary inside R_h.

where h_n is a p-periodic sequence of stocking quotas, and p denotes the minimal period. Observe that some consecutive values of the stocking sequence can be zero; however, it is natural to assume $\sum_{j=0}^{p-1} h_j > 0$. As in the constant case, we associate Eq. (3.1) with a p-periodic sequence of two dimensional maps that we use in the sequel, namely $\{T_j, j = 0, 1, \ldots, p-1\}$, where $T_j(x, y) = (y, yf(x) + h_j)$. It is obvious that if we replace h by h_j in Theorem 2.1, then R_{h_j} forms a compact and invariant region for the individual map T_j, which enables us to build a suitable machinery for establishing the existence of a periodic solution. It is convenient now to develop the notations of the previous section so it can suit the periodic case. We denote the line segments that form the boundary of R_{h_j} by $\Gamma_{j,i}, i = 0, \ldots, 4$ where $\Gamma_{j,i}$ corresponds to Γ_i in the autonomous case, and that is associated with the individual map T_j. Also, the constant c_h in Proposition 2.4 will be replaced by c_{h_j} and that is associated with the individual map T_j.

3.1 Existence of a periodic solution

We start by establishing a compact invariant for Eq. (3.1). Define

$$h_m := \max\{h_0, h_1, \cdots, h_{p-1}\} \quad \text{and} \quad c_m := \max\{c_{h_j} : \ j = 0, \ldots, p-1\},$$

where c_{h_j} as taken in Proposition 2.4, i.e., $c_{h_j} = \frac{1}{b} \sup_t G_{h_j}(t)$, then use h_m and c_m to define the region R_{h_m} as in the paragraph preceding Theorem 2.1. Now, we have the following result:

Lemma 3.1. *Consider Eq. (3.1) together with the associated p-periodic sequence of maps $\{T_j\}$. Each of the following holds true:*

(i) We have $R_{h_i} \subseteq R_{h_j}$ Whenever $h_i \leq h_j$.

(ii) R_{h_m} is a compact invariant for each individual map T_j.

(iii) R_{h_m} is a compact invariant for the map $\hat{T} := T_{p-1} \circ T_{p-2} \circ \cdots \circ T_0$.

Proof. (i) When $h_i \leq h_j$, we obtain $G_{h_i}(t) \leq G_{h_j}(t)$ for all $t \geq 0$. Thus, $c_{h_i} \leq c_{h_j}$ and the result becomes obvious from Proposition 2.4 and the geometric structure of the regions R_{h_i} and R_{h_j}. To prove (ii), let $(x, y) \in R_{h_m}$, we show that $T_j(x, y) \in R_{h_m}$. Since

$$T_j(x, y) = (y, yf(x) + h_j) = (y, yf(x) + h_m) - (0, h_m - h_j) = T_m(x, y) - (0, h_m - h_j),$$

the first component of $T_j(x, y)$ is the same as the first component of $T_m(x, y)$ and the second component of $T_j(x, y)$ is lower than the second component of $T_m(x, y)$. Now, the fact that $T_m(x, y) \in R_{h_m}$ and the geometric structure of R_{h_m} assures that $T_j(x, y) \in R_{h_m}$. Finally, (iii) follows from (ii). □

Periodic stocking (or harvesting) has the effect of forcing population cycles to evolve and become multiples of the stocking/harvesting period as we show in the following result, which is more general than Eq. (3.1).

Theorem 3.1. *Consider the general difference equation $x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k})$ with p-periodic stocking (or harvesting). If a periodic solution exists, then the period is a multiple of p.*

Proof. The proof is by contradiction; suppose that we have a r-periodic solution of the equation $x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}) + h_n$ for some r that is not a multiple of p. Then the greatest common divisor between r and p ($d := \gcd(r, p)$) is not p. Define the maps $F_i := F + h_i,\ i = 0, 1, \ldots, p-1$, then for each $0 \leq i \leq d-1$, the maps $\{F_{kd+i}, k = 0, 1, \ldots, \frac{p}{d} - 1\}$ must agree on the point $X_i := (x_i, x_{i-1}, \ldots, x_{i-k})$, where the components $x_{i-k}, x_{i+1-k}, \ldots, x_i$ are consecutive elements of the p-periodic solution. This implies

$$h_i = h_{d+i} = h_{2d+i} = \cdots = h_{(\frac{p}{d}-1)d+i}$$

for all $i = 0, 1, \ldots, d-1$, which contradicts the minimality of the period of the p-periodic difference equation. □

Theorem (3.1) shows that Eq. (3.1) has no equilibrium solutions, and therefore, our previous notion of characterizing oscillatory solutions based on the oscillations about $y = x$ is the valid one here. Thus, solutions of Eq. (3.1) are oscillatory about $y = x$ because they cannot be monotonic. Although it is natural for fluctuations in the environment to create fluctuations in the population, we find it appropriate here to connect the loosely-defined term "fluctuation" with the mathematically well-defined term "oscillation." Next, we use the Brouwer fixed theorem [30] (page 51) to prove the existence of a periodic solution of Eq. (3.1):

Theorem 3.2 (Brouwer Fixed-Point Theorem)**.** *Let M be a a nonempty, convex and compact subset of $\mathbb{R}^n$. If $T : M \to M$ is continuous, then T has a fixed point in M.*

Theorem 3.3. *The p-periodic difference equation in Eq. (3.1) has a p-periodic solution.*

Proof. Consider the map $\hat{T} := T_{p-1} \circ T_{p-2} \circ \cdots \circ T_0$, then using Lemma 3.1, we obtain $\hat{T} : R_{h_m} \to R_{h_m}$. Furthermore, R_{h_m} is nonempty, compact and obviously convex. So, by Theorem 3.2, $\hat{T}$ has a fixed point in R_{h_m}. This fixed point establishes a periodic solution of Eq. (3.1) with minimal period that divides p; however, Theorem 3.1 shows that the period must be p. □

3.2 Global attractivity of the periodic solution when $p = 2$

Consider the periodicity of Eq. (4.1) to be $p = 2$ and suppose $h_0 + h_1 \neq 0$. We partition solutions of Eq. (4.1) into two subsequences, the one with even indices $\{x_{2n}\}$ and the one with odd indices $\{x_{2n}\}$. Thus, we have

$$x_{2n+1} = x_{2n} f(x_{2n-1}) + h_0 \qquad \text{and} \qquad x_{2n+2} = x_{2n+1} f(x_{2n}) + h_1. \tag{3.2}$$

Since solutions are bounded, we define

$$\liminf\{x_{2n+i}\} = I_i \quad \text{and} \quad \limsup\{x_{2n+i}\} = S_i, \quad i = 0, 1. \tag{3.3}$$

Now, the second iterate of Eqs. (3.2) gives us

$$x_{2n+2} = x_{2n} f(x_{2n}) f(x_{2n-1}) + h_0 f(x_{2n}) + h_1 \tag{3.4}$$

$$x_{2n+3} = x_{2n+1} f(x_{2n+1}) f(x_{2n}) + h_1 f(x_{2n+1}) + h_0. \tag{3.5}$$

Use the fact that $f(t)$ is decreasing and $tf(t)$ is increasing in Eq. (3.4) to obtain

$$S_0 \leq S_0 f(S_0) f(I_1) + h_0 f(I_0) + h_1 \tag{3.6}$$

$$I_0 \geq I_0 f(I_0) f(S_1) + h_0 f(S_0) + h_1. \tag{3.7}$$

Also, Eq. (3.5) gives us

$$S_1 \le S_1 f(S_1) f(I_0) + h_1 f(I_1) + h_0 \tag{3.8}$$

$$I_1 \ge I_1 f(I_1) f(S_0) + h_1 f(S_1) + h_0. \tag{3.9}$$

Multiply Inequality (3.6) by I_0 and Inequality (3.7) by S_0 to obtain

$$S_0 I_0 f(I_0) f(S_1) + S_0(h_0 f(S_0) + h_1) \le I_0 S_0 f(S_0) f(I_1) + I_0(h_0 f(I_0) + h_1) \tag{3.10}$$

Sine $I_0(h_0 f(I_0) + h_1) \le S_0(h_0 f(S_0) + h_1)$, we obtain

$$f(I_0) f(S_1) \le f(S_0) f(I_1). \tag{3.11}$$

Also, multiply Inequality (3.9) by I_1 and Inequality (3.9) by S_1 to obtain

$$S_1 I_1 f(I_1) f(S_0) + S_1(h_1 f(S_1) + h_0) \le I_1 S_1 f(S_1) f(I_0) + I_1(h_1 f(I_1) + h_0) \tag{3.12}$$

Sine $I_1(h_1 f(I_1) + h_0) \le S_1(h_1 f(S_1) + h_0)$, we obtain

$$f(I_1) f(S_0) \le f(I_0) f(S_1). \tag{3.13}$$

Using inequalities (3.11) and (3.13), we obtain the following result:

Lemma 3.2. *Consider I_0, I_1, S_0, S_1 as defined in Eqs. (3.3), then $f(I_0)f(S_1) = f(I_1)f(S_0)$.*

Next, we give the following result:

Theorem 3.4. *For $p = 2$, the 2-periodic solution of Eq. (3.1) is a global attractor.*

Proof. Use the result of Lemma 3.2 in Inequality (3.10) to obtain

$$S_0(h_0 f(S_0) + h_1) \le I_0(h_0 f(I_0) + h_1).$$

Since $g(t) = t(h_0 f(t) + h_1)$ is increasing and $I_0 \le S_0$, we must have $I_0 = S_0$. Similarly, use the result of Lemma 3.2 in Inequality (3.12) to obtain

$$S_1(h_1 f(S_1) + h_0) \le I_1(h_1 f(I_1) + h_0),$$

and consequently $S_1 = I_1$. Hence, $I_i = S_i,\ i = 0, 1$ and the proof is complete. □

Remark 3.1. *Observe that the approach of this section proves not only the global attractivity of the p-periodic solution but also the existence; however, Theorem 3.1 is still significant here because it proves the minimality of the period. Also, establishing the compact invariant region in Lemma 3.1 deserves embracing regardless of the globally attractivity of the periodic solution. Finally, proving the global attractivity for general p will be the topic of some future work.*

4 Peilou's equation with stocking

As an illustrative example to our results, we consider the function f in Eq. (3.1) to be $f(t) = \frac{bt}{1+t}$. It is worth mentioning that in the absence of stocking, Pielou ([19], page 80) suggested taking $f(x_{n-m}) = \frac{\mu K}{K+(\mu-1)x_{n-m}}$ to account for certain fluctuating populations, which cannot be modeled by Beverton-Holt equation. So, here we are dealing with the dimensionless Pielou's equation $y_{n+1} = \frac{by_n}{1+y_{n-1}}$, which takes the following form after forcing stocking:

$$y_{n+1} = \frac{by_n}{1+y_{n-1}} + h_n. \tag{4.1}$$

When $h_n = 0$, Eq. (4.1) has the positive equilibrium $x_{2,0} = b-1$ which is globally asymptotically stable. When $h_n = h > 0$, $x_{2,h} = \frac{1}{2}(b+h-1) + \frac{1}{2}\sqrt{(b+h-1)^2+h}$ inherits the global asymptotic stability of $x_{2,0}$ as shown in [18]. Now, consider $\{h_n\}$ to be 2-periodic. The 2-periodic solution $\{\bar{x}, \bar{y}\}$ is the solution of the system of equations

$$(y-h_0)(1+y) = bx \quad \text{and} \quad (x-h_1)(1+x) = by.$$

However, the solution is not simple to write explicitly, and therefore, we proceed by choosing $h_0 = b + 1/(b-1)$ and $h_1 = b$. In this case, the 2-periodic solution $\{\bar{x}, \bar{y}\}$ is given by

$$\bar{x} = b - 1 + \frac{b^{\frac{3}{2}}}{\sqrt{b-1}} \quad \text{and} \quad \bar{y} = b + \frac{1}{b-1} + \sqrt{b(b-1)}.$$

Figure 3 shows numerically that solutions are attracted to the 2-periodic cycle.

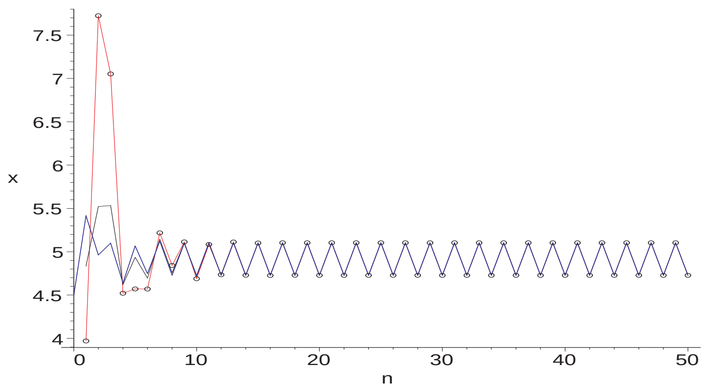

Figure 3: This graph shows the stable 2-cycle for the 2-periodic equation in Eq. (4.1) when $h_0 = b + \frac{1}{b-1}$ and $h_1 = b$, where b is fixed at $\frac{5}{2}$.

Another interesting notion that can be observed here is the resonance of solutions of Eq. (4.1). The arithmetic average of the globally attracting 2-periodic solution is

$$x_{sv} := \frac{1}{2}(\bar{x} + \bar{y}) = \frac{1}{2}\left(2b - 1 + \frac{1}{b-1} + \frac{b^{\frac{3}{2}}}{\sqrt{b-1}} + \sqrt{b(b-1)}\right).$$

On the other hand, when we take the constant stocking $h = \frac{1}{2}(h_0 + h_1) = b + \frac{1}{2(b-1)}$, we obtain the globally attracting equilibrium

$$\bar{x} = \frac{4b^2 - 6b + 3 + \sqrt{16b^4 - 32b^3 + 28b^2 - 12b + 1}}{4(b-1)}.$$

Figure 4 shows that $x_{av} > \bar{x}$.

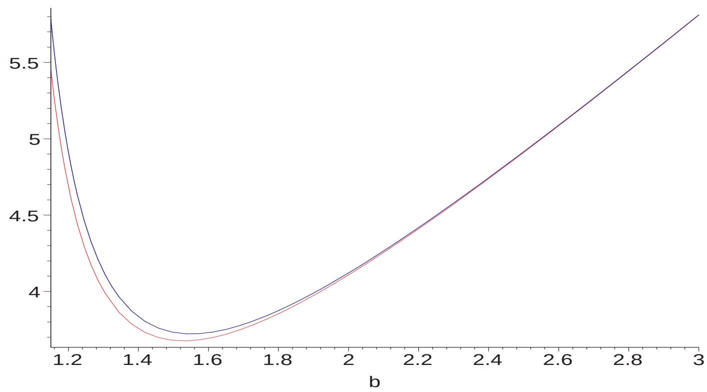

Figure 4: This graph shows the average of the attracting 2-cycle (blue color) in contrast with the equilibrium that results from constant stoking equals the average of h_0 and h_1, where $h_0 = b + 1/(b-1)$ and $h_1 = b$.

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