Harvesting and stocking in contest competition models: Open problems and conjectures *

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Abstract

In this survey, we consider a class of difference equations that encompasses contest competition models of one species. We allow short delay in the recruitment, and force several harvesting/stocking strategies. We provide a summary of some recent results related to the dynamics of this class of models, and give some open problems and conjectures that are worth exposing to the audience of difference equations.

Key words: Contest competition, harvesting, stocking, persistence, global stability.
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1 Introduction

Difference equations of the form $x_{n+1} = x_n f(x_n)$, $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$ are used in modeling certain populations with non-overlapping generations, where x_n is the population size at a discrete time unit n, and the function $f(x)$ denotes the per-capita growth rate [7, 18, 19, 25]. When successful individuals in certain species get all requirements while unsuccessful ones get insufficient for survival or reproduction, the model is known as contest competition [30]. To understand the mechanistic basis of various discrete population models, we refer the interested reader to a recent study by Anazawa [6] and to the study of Lomnicki [17]. From a mathematical perspective, this motivates us to consider a class of functions that encompass contest competition models, and therefore, we proceed with the following assumptions on f.

(A1) $f \in C^1([0, \infty))$ and f is decreasing;

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(A2) $f(0) = b > 1$;

(A3) $tf(t)$ is increasing and bounded by a constant M.

When a time lag occurs in the recruitment, one can consider the difference equation with delay $x_{n+1} = x_n f(x_{n-k})$, where k is a positive integer [15]. The general effect of time lag on stability of steady states and population oscillations has been investigated in the literature [20, 22, 29].

In this survey, we are interested in controlling the delay effect, and thus, we consider the difference equation

$$x_{n+1} = x_n f(x_{n-k}), \; k = 0, 1, \tag{1.1}$$

then we investigate the dynamics with the effect of harvesting or stocking. In general, forcing a stocking or harvesting term on Eq. (1.1) leads to the equation

$$x_{n+1} = x_n f(x_{n-k}) \pm H_n(x_n, x_{n-1}), \quad k = 0, 1. \tag{1.2}$$

Eq. (1.1) has two equilibrium solutions, namely zero and $\bar{x} := f^{-1}(1)$. The existence of the positive equilibrium $\bar{x}$ is assured by assumption (A2) on f. Now, some of the general questions that worth investigating in Eq. (1.2) are related to the effect of stocking, effect of harvesting, optimal harvesting or maximum sustainable yield, persistence and the persisting set, ordering the harvesting quotas in periodic harvesting, "best" harvesting strategy,...etc. However, keeping H_n in its general form gives untraceable dynamics. In the sequel of this survey, we consider special cases of H_n, summarize some of the results established in the literature [1–5] and expose some open problems and conjectures. For each choice of H_n that we take, we find it convenient to discuss the dynamics for $k = 0$ and $k = 1$ as separate cases.

2 Constant yield harvesting/stocking

In this section, we discuss the dynamics of Eq. (1.2) when $H_n(x_n, x_{n-1})$ is taken to be constant. When the constant is negative, the strategy is known as constat catch or constant yield strategy [8, 11, 26]. On the other hand, the constant is taken positive when the species modeled by Eq. (1.1) is affected by stocking due to refuge, immigration,...etc [21, 28].

2.1 No time lag ($k = 0$)

Consider the equation

$$x_{n+1} = x_n f(x_n) + h, \quad h \in \mathbb{R}. \tag{2.1}$$

At $h = 0$, we have the two equilibrium solutions $\bar{x}_{1,0} = 0$ and $\bar{x}_{2,0} = f^{-1}(1)$. When h is positive (stocking), $\bar{x}_{1,h}$ shifts below $f^{-1}(1)$ while $\bar{x}_{2,h}$ shifts above $f^{-1}(1)$. Thus, $\bar{x}_{1,h}$ is beyond our interest and we are left with the positive equilibrium $\bar{x}_{2,h}$, which is increasing in h. Using a simple cobweb diagram, we observe that $\bar{x}_{2,h}$ is globally attractive.

When h is negative (harvesting), $\bar{x}_{1,h}$ shifts upward and $\bar{x}_{2,h}$ shifts downward till they collide at a maximum harvesting level

$$h_{\max} := x(f(x) - 1), \quad 0 \le x \le f^{-1}(1). \tag{2.2}$$

A harvesting level beyond $h_{\max}$ leads to a total collapse of the population, while $0 < -h < h_{\max}$ assures the survival of all initial populations that are larger than or equal to the small equilibrium $\bar{x}_{1,h}$. Again, a cobweb diagram can be used to show that $(\bar{x}_{1,h}, \infty)$ is the basin of attraction of $\bar{x}_{2,h}$.

2.2 One-unit time lag ($k = 1$)

Consider the equation

$$x_{n+1} = x_n f(x_{n-1}) \pm h, \tag{2.3}$$

where h is a positive parameter representing a constant stocking or harvesting quota. We have the same equilibrium solutions as in the case $k = 0$; however, the dynamics becomes a bit more challenging. At $h = 0$, the one-unit time lag does not change the bounded character of solutions, but monotonic convergence changes into oscillatory convergence. Boundedness of solutions and convergence to $\bar{x}$ can be found in [23]. Also, the global stability of $\bar{x}$ can be extracted from [16], which investigates the global stability of models that encompass the equation $x_{n+1} = x_n f(x_{n-1})$. One way to show the oscillating nature of solutions is by setting a new coordinate system at the positive equilibrium $\bar{x}$, then use the map

$$T_0 : \mathbb{R}^{+2} \to \mathbb{R}^{+2} \qquad \text{defined by} \qquad T_0(x, y) = (y, yf(x)) \tag{2.4}$$

to show that T_0 rotates the quadrants of the new coordinate system clockwise [1].

Next, we proceed by taking stocking and harvesting as separate cases.
The stocking case: In this case, Eq. (2.3) becomes

$$x_{n+1} = x_n f(x_{n-1}) + h, \; h > 0. \tag{2.5}$$

Solutions of Eq. (2.5) are bounded as we can see from the fact that $x_{n+1} \ge 0$ and

$$x_{n+2} = x_n f(x_n) f(x_{n-1}) + hf(x_n) + h \le Mb + hb + h,$$

where M is the bound given in Assumption (A3). Moreover, the positive equilibrium $\bar{x}_{2,h}$ preserves its global attractiveness. Indeed, we can use [2, 22] to extract the following result:

Lemma 2.1. *Let $h > 0$ in Eq. (2.3). Solutions are oscillatory and*

$$\lim_{n\to\infty} x_n = \bar{x}_{2,h}.$$

Next, modify the map T_0 in Eq. (2.4) to be $T_h(x, y) = (y, yf(x) + h)$, then T_h can be used to portray solutions of Eq. (2.3) with $h > 0$ as orbits in the positive quadrant. A region R_h forms an invariance of Eq. (2.3) if $T_h(R_h) \subseteq R_h$. It was shown in [2] that we obtain a bounded and invariant region by connecting the points $(0, 0), (0, h), (c_h, bc_h + h)$, $(bc_h + h, bc_h + h), (bc_h + h, 0)$ and $(0, 0)$, respectively with line segments, where c_h is taken to be $\frac{1}{b} \sup_t (bt + h)f(t)$.

The harvesting case: In this case, we deal with the equation

$$x_{n+1} = x_n f(x_{n-1}) - h, \tag{2.6}$$

where $h > 0$ is a parameter representing a harvesting quota. This equation was investigated in [1], and therefore, we refer the interested reader to [1] for more elaborated details. Before we proceed, we give a formal definition of persistence and strong persistence. A solution of Eq. (1.2) is called persistent if the corresponding population survives indefinitely. We call a persistent solution strongly persistent if $\liminf x_n > 0$. A set $\mathcal{D} := \{(x, y) : (x, y) \in \mathbb{R}^{+2}\}$ is persistent if each solution of Eq. (1.2) with $(x_{-1}, x_0) \in \mathcal{D}$ is persistent. In Eq. (2.6), we use $\mathcal{D}_h$ to denote the largest persisting set at the harvesting level h. From Eq. (2.6), persistence implies $x_n \geq \frac{h}{f(x_{n-1})}$, and consequently $x_n \geq \frac{h}{b}$. Thus, persistence implies strong persistence.

Theorem 2.1. *[1] Consider Eq. (2.6), and define h_{max} as given in Eq. (2.2). Each of the following holds true:*

(i) Persistent solutions are bounded and strongly persistent.

(ii) If $h > h_{max}$, then $\mathcal{D}_h$ is empty.

(iii) If $h = h_{max}$, then all elements of $\mathcal{D}_h$ are attracted to $\bar{x}_{h_{max}} := \bar{x}_{1,h} = \bar{x}_{2,h}$.

Computer simulations show that $\mathcal{D}_h$ shrinks as h increases, which is in accord with the fact provided about the basin of attraction of $\bar{x}_{2,h}$ in the absence of time lag; however, a mathematical proof is missing in case of Eq. (2.6). We formalize the observation in the following conjecture.

Conjecture 2.1. *Consider Eq. (2.6). The set $\mathcal{D}_h$ is decreasing in h, i.e., if $h_1 \leq h_2$, then $\mathcal{D}_{h_2} \subseteq \mathcal{D}_{h_1}$.*

The equilibrium $\bar{x}_{1,h}$ is a saddle for all $0 \leq h \leq h_{max}$. At $h = 0$, the stable manifold does not appear in the positive quadrant, and therefore, $\bar{x}_{1,h}$ can be ignored. However, when $h > 0$, the stable manifold of $\bar{x}_{1,h}$ becomes in the persistent set, which makes the dynamics of Eq. (2.6) interesting and challenging at the same time. In [1], a comparison principle was used to show that persistent solutions are eventually larger than or equal to $\bar{x}_{1,h}$ as we show in the following three extracted results:

Theorem 2.2. *Consider Eq. (2.6), and suppose there are two sequences α_n and β_n such that $\alpha_n \leq x_n \leq \beta_n$ for all $n \geq -1$. Define $f_{n+1}(x) = F(x, \alpha_n)$ and $g_{n+1}(x) = F(x, \beta_n)$, then we obtain*

$$g_n g_{n-1} \cdots g_0(x_0) \leq x_{n+1} \leq f_n f_{n-1} \cdots f_0(x_0).$$

Theorem 2.3. *Consider the orbit of $t_0 := \frac{b}{h}$ in the equation $t_{n+1} = \frac{(t_n+h)}{f(t_n)}$, then t_n monotonic and converges to $\bar{x}_{1,h}$. Furthermore, any persistent solution $\{x_n\}_{-1}^{\infty}$ of Eq. (2.6) satisfies $x_n \geq t_n$ for all $n \in \mathbb{N}$.*

Theorem 2.4. *A persistent solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (2.6) satisfies*

$$\bar{x}_{1,h} \leq \liminf x_n \leq \limsup x_n \leq x_{2,h}^*,$$

where $x_{2,h}^*$ is the largest fixed point of the function $g(t) = h\dfrac{f(t)+1}{f(t)f(\bar{x}_{1,h})-1}$ in the interval $\left[0, f^{-1}\left(1/f(\bar{x}_{1,h})\right)\right)$.

We can force harvesting on Pielou's equation [14, 24] and use it as a "toy model" to represent Eq. (2.6). Indeed, we have

$$y_{n+1} = \frac{K\mu y_n}{K+(\mu-1)y_{n-1}} - h^*, \quad \mu > 1, K > 0, h^* > 0. \tag{2.7}$$

Let $x_{n-1} := \frac{(\mu-1)}{K} y_{n-1}$, $h := \frac{(\mu-1)}{K} h^*$ and $b := \mu$, we obtain

$$x_{n+1} = \frac{bx_n}{1+x_{n-1}} - h, \quad b > 1, h > 0. \tag{2.8}$$

Thus $\frac{b}{1+t} = f(t)$ in Eq. (2.6). The dynamics of Eq. (2.8) was investigated in [1]. Nevertheless, we extract the following facts and provide some questions that worth further investigation.

At $h = 1$ and $b \geq 4$, Eq. (2.8) is related to Lyness equation [9, 10, 13] and has the invariants

$$\mathcal{I}_b(x,y) := \left(1+\frac{b}{1+x}\right)\left(1+\frac{b}{1+y}\right)(1+x+y) = \mathcal{I}_b(x_{-1}, x_0). \tag{2.9}$$

In this case, the persistence set $\mathcal{D}_1$ can be found explicitly. Indeed, $(x, y) \in \mathcal{D}_1$ if and only if

$$2 + (b+1)^2 - \bar{x}_2(b-4) \leq \mathcal{I}_b(x,y) \leq 2 + (b+1)^2 - \bar{x}_1(b-4).$$

When $h = 1$ and $b > 2(1+\sqrt{2})$, an 8-periodic solution of Eq. (2.8) was found and used to define a trapping region for Eq. (2.8) with $0 < h < 1$. Now, we pose the following problems:

Conjecture 2.2. *Consider Eq. (2.8) with $0 < h < 1$. All persistent solutions larger than $\bar{x}_{1,h}$ are attracted to $\bar{x}_{2,h}$.*

Open Problem 2.1. *Consider Eq. (2.8) with $0 < h < 1$. Show that $\mathcal{D}_1 \subseteq \mathcal{D}_h$.*

Open Problem 2.2. *Consider Eq. (2.8) with $h > 1$. Characterize $\mathcal{D}_h$. Is $\mathcal{D}_h$ closed? Is $\mathcal{D}_h$ connected?*

3 Periodic Harvesting/stocking

Harvesting or stocking can be controlled or regulated to prevent species extinction or to improve the total yield over a period of time. However, the question how to regulate harvesting/stocking is widely open for research and debate [8, 11, 27, 31]. De Klerk and Gatto considered a continuous multi-cohort Beverton-Holt model [12] and argued that adopting a periodic fishing strategy instead of a constant effort strategy is worthwhile when there is a significant economy of scale, and when older fish are much more valuable than younger ones. AlSharawi and Rhouma considered the discrete Beverton-Holt model [3] and investigated the dynamics of the model with the effect of several harvesting strategies. The chart given in the conclusion of [3] gives a theoretical suggestion for considering each strategy.

In this section, we force periodic stocking/harvesting on Eq. (1.1), then discuss the dynamics when $k = 0$ and $k = 1$. The next result shows that population cycles evolve under periodic stocking/harvesting and become multiples of the stocking/harvesting period.

Theorem 3.1. *[2] If there exists a r-periodic solution of the p-periodic difference equation*

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}) \pm h_n, \quad h_{n+p} = h_n,$$

then r is a multiple of p.

3.1 No time lag ($k = 0$)

Consider the p-periodic equation

$$x_{n+1} = x_n f(x_n) \pm h_n,\ h_n \geq 0, \tag{3.1}$$

where $\{h_n\}$ is a p-periodic sequence representing periodic stocking or harvesting. Although it is possible not to have stocking in some seasons ($h_j = 0$ for some j), we want to avoid reducing Eq. (3.2) to $x_{n+1} = x_n f(x_n)$, and therefore, we assume $\sum h_n > 0$.

The stocking case: In this case, Eq. (3.1) becomes

$$x_{n+1} = x_n f(x_n) + h_n, \tag{3.2}$$

where $h_n \geq 0$ and $h_{n+p} = h_n$ for all $n \in \mathbb{N}$. Define the maps $g_n(x) = xf(x) + h_n$. Since each g_n is an upward shift of $xf(x)$, the period of any periodic solution has to be a multiple of p [2,4]. Define the p-fold functions $F_j := g_{p+j-1} \circ g_{p+j-2} \circ \cdots g_j$, then for each $j = 0, 1, \dots, p-1$, F_j is increasing and bounded with $F_j(0) > 0$. Thus, F_j has a unique positive fixed point, say $\bar{x}_{j,h_n}$. Furthermore,

$$\lim_{n\to\infty} x_{np+j} = \bar{x}_{j,h_n}.$$

Now, $\{\bar{x}_{0,h_n}, \bar{x}_{1,h_n}, \cdots, \bar{x}_{p-1,h_n}\}$ is a p-periodic solution of Eq. (3.2), which is a global attractor. Let $\{h_n\}$ be a p-periodic sequence of stocking quotas. Define

$$h_{av} := \frac{1}{p}\sum_{j=0}^{p-1} h_j.$$

Now, sum Eq. (**??**) over the periodic attractor to obtain

$$\sum_{j=0}^{p-1} \bar{x}_j = \left(\sum_{j=0}^{p-1} \bar{x}_j f(\bar{x}_j) + h_j\right).$$

If $y = tf(t)$ is concave, then we can use Jensen's inequality to conclude that

$$\bar{x}_{av} \leq \bar{x}_{av} f(\bar{x}_{av}) + h_{av}.$$

Thus, $\bar{x}_{av} \leq \bar{x}_{2,h_{av}}$, where $x_{2,h_{av}}$ is the globally stable equilibrium at a constant stocking level $h = h_{av}$. In this case, populations attenuate under periodic stocking. However, a more ambiguous notion that needs deep investigation is the following: How does the order of the stocking quotas affect the population average? We formulate this question in the following open problem:

Open Problem 3.1. *Let $\{h_n\}$ be a p-periodic sequence of stocking quotas, and let $\{\hat{h}_n\}$ be a permutation of $\{h_n\}$. Define x_{av} and $\hat{x}_{av}$ to be the average of the global attractors associated with $\{h_n\}$ and $\{\hat{h}_n\}$, respectively. How does x_{av} relate to $\hat{x}_{av}$?*

The harvesting case: In this case, Eq. (3.1) becomes

$$x_{n+1} = x_n f(x_n) - h_n, \tag{3.3}$$

where $h_n \geq 0$, $\sum h_j > 0$ and $h_{n+p} = h_n$ for all $n \in \mathbb{N}$. Obviously, sufficiently large values of h_n lead to a total collapse in the population. Thus, finding a maximum sustainable yield (MSY) is an issue of particular interest here. The MSY can be found from the following constraints

$$F_j(x) = x \quad \text{and} \quad F_j'(x) = 1 \tag{3.4}$$

for all $j = 0, 1, \ldots, p-1$. As an illustrative example, we discuss the Beverton-Holt model [7].

Example 3.1. *Consider the Beverton-Holt model with 2-periodic harvesting given by*

$$x_{n+1} = \frac{K\mu x_n}{K + (\mu - 1)x_n} - h_n, \tag{3.5}$$

where $h_{n+2} = h_n$ for all $n \in \mathbb{N}$ and $h_0, h_1 > 0$.

Based on the constraints in Eqs. (3.4), we can eliminate x and obtain a relationship between h_0 and h_1. Indeed, we obtain

$$K^2 - K\beta(h_0 + h_1) + h_0 h_1 = 0, \quad \beta = \frac{\mu+1}{\mu-1},$$

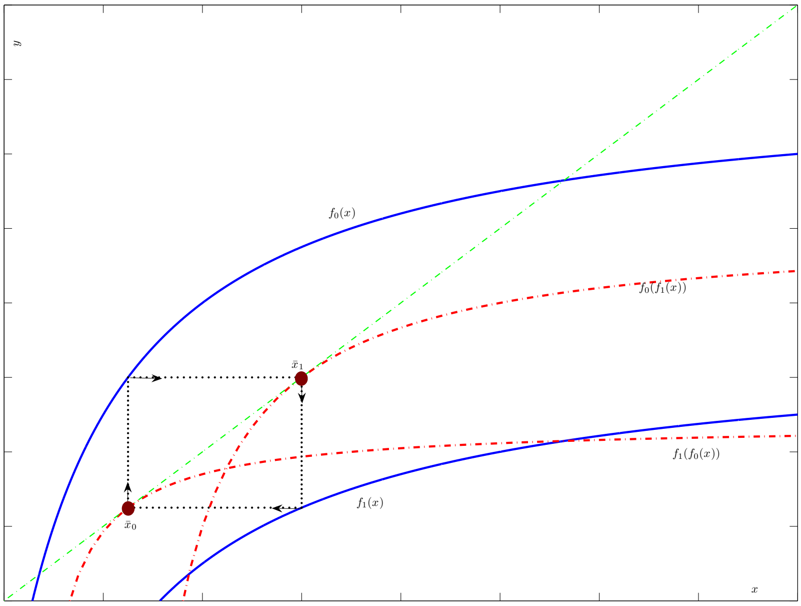

Figure 1: This figure shows the curves of $f_0(x), f_1(x), f_1(f_0(x))$ and $f_0(f_1(x))$ together with the 2-cycle $\{\bar{x}_0, \bar{x}_1\}$. The parameters are fixed as $K = 4, \mu = 9, h_0 = 1$ and $h_1 = \frac{11}{4}$.

or equivalently

$$h_1 = \frac{K(K - \beta h_0)}{K\beta - h_0}, \quad h_0 < \frac{K}{\beta}.$$

We use the relationship between h_0 *and* h_1 *to find*

$$\bar{x}_0 = \frac{K(K + h_0)}{K(\mu + 1) - h_0(\mu - 1)} \quad \text{and} \quad \bar{x}_1 = \frac{1}{2}(K - h_0).$$

At $h_0 = \frac{(\sqrt{\mu}-1)}{\sqrt{\mu}+1}K$, *we obtain* $h_0 = h_1$ *and* $\bar{x}_0 = \bar{x}_1$, *which is the constant harvesting case. Observe that a swap of* h_0 *and* h_1 *leads to a swap of* $\bar{x}_0$ *and* $\bar{x}_1$, *which seems to be of little mathematical effect, but in fact, it has a dramatic effect on low level populations. When* $h_0 < h_1$, *we have* $\bar{x}_0 < \bar{x}_1$ *and populations in* $[\bar{x}_0, \infty)$ *persist. On the other hand,* $h_0 > h_1$ *implies* $\bar{x}_0 > \bar{x}_1$ *and* $[\bar{x}_0, \infty)$ *is the persistent set. Therefore, one can investigate the advantage of having* $0 \leq h_0 \leq h_1 \leq h_{max}$ *at all times. See Figure 1 for an illustration.*

The facts discussed in Example 3.1 motivate investigating the following open problems. We use $\mathcal{D}(h_0, h_1, \ldots, h_{p-1})$ to denote the persistent set of Eq. (3.5).

Open Problem 3.2. *Let* $\{h_n\}$ *be a p-periodic sequence of harvesting quotas in Eq. (3.5) that give a nonempty persistent set* $\mathcal{D}(h_0, h_1, \ldots, h_{p-1})$. *Let* $\{\hat{h}_n\}$ *be a permutation of* $\{h_n\}$. *Define* x_{av} *and* $\hat{x}_{av}$ *to be the average of the attractors associated with* $\{h_n\}$ *and* $\{\hat{h}_n\}$, *respectively. How does* x_{av} *relate to* $\hat{x}_{av}$*?*

Open Problem 3.3. *Let $\{h_n\}$ be a fixed p-periodic sequence of harvesting quotas that give a nonempty persistent set $\mathcal{D}(h_0, h_1, \ldots, h_{p-1})$. Which permutation of $\{h_n\}$ gives the largest persistent set?*

Open Problem 3.4. *What happens to the invariants given in Eq. (2.9) when $h_n = 1 \pm \epsilon_n$.?*

3.2 One-unit time lag ($k = 1$)

Here, we have the p-periodic difference equation with delay

$$x_{n+1} = x_n f(x_{n-1}) \pm h_n, \quad h_{n+p} = h_n. \tag{3.6}$$

The stocking case: Consider the equation

$$x_{n+1} = x_n f(x_{n-1}) + h_n, \tag{3.7}$$

where $h_n \geq 0$ is a p-periodic sequence representing stocking quotas ($\sum h_j > 0$). Eq. (3.7) was investigated by the author in [2]. Define the two dimensional maps $T_j(x, y) = (y, yf(x)+h_j)$, then the iterates of the p-periodic sequence of maps $T_j : \ j = 0, 1, \ldots, p-1$ portray the dynamics of Eq. (3.7) in the positive quadrant. As in the paragraph proceeding Lemma 2.1, we define the region R_{h_j} for each individual map T_j to obtain a compact invariant for each map T_j. However, we need a compact invariant for the p-fold map $T = T_{p-1} \circ T_{p-2} \circ \cdots \circ T_0$. It was shown in [2] that $h_i \leq h_j$ implies $R_{h_i} \subseteq R_{h_j}$, which suggests defining one invariant for all maps T_j. Indeed, consider $h_m := \max_j\{h_j, j = 0, 1, \ldots, p-1\}$ and

$$c_m := \max_j\{\frac{1}{b} \sup_{t\geq 0}(bt + h_j)f(t) : \ j = 0, 1, \ldots, p-1\},$$

then use h_m and c_m to define a region R_{h_m} as in the paragraph proceeding Lemma 2.1. The region R_{h_m} is a compact invariant for each map T_j, and consequently it is a compact invariant for the p-fold map T. Using Brouwer Fixed-Point Theorem, T has a fixed point in R_{h_m}. Based on Theorem 3.1, we obtain a p-periodic solution of Eq. (3.7). Now, the following theorem summarizes the main result obtained in [2].

Theorem 3.2. *The p-periodic difference equation in Eq. (3.7) has a p-periodic solution. Furthermore, the p-periodic solution is globally attracting when $p = 2$.*

Conjecture 3.1. *The p-periodic solution of Eq. (3.7) obtained in Theorem 3.2 is globally attracting for all $p > 2$.*

Finally, after verifying this conjecture, then Problem 3.1 can be investigated for Eq. (3.7).

The harvesting case: In this case, Eq. (3.6) takes the form

$$x_{n+1} = x_n f(x_{n-1}) - h_n, \quad h_n \geq 0. \tag{3.8}$$

To the best of my knowledge, Eq. (3.8) has not been studied yet. Define

$$h_{min} = \min\{h_0, h_1, \dots, h_{p-1}\}.$$

As in Theorem 2.1, because

$$x_{n+2} = x_n f(x_n) f(x_{n-1}) - h_n f(x_n) - h_{n+1} \leq Mb - h_{min}(b+1)$$

and persistent solutions satisfy $x_n \geq \frac{h_{min}}{b}$, then persistent solutions are bounded and strongly persistent. Furthermore, we give the following:

Proposition 3.1. *Consider h_{max} as defined in Eq. (2.2). If $h_n > h_{max}$ for all $n = 0, 1, \dots, p-1$, then the persistent set of Eq. (3.8) is empty.*

Proof. Start with (x_{-1}, x_0) such that $x_{-1} \geq x_0$, then $x_1 = x_0 f(x_{-1}) - h_0 < x_0 f(x_0) - h_0 < x_0$. By induction, we obtain a decreasing and bounded sequence that must converge to a value say α. For $j = 0, 1, \dots, p-1$, we have

$$x_{np+j} = x_{np+j-1} f(x_{np+j-2}) - h_{j-1},$$

which implies α is a fixed point of this equation. Since $h_{j-1} > h_{\max}$, we obtain a contradiction. Next, start with (x_{-1}, x_0) such that $x_{-1} < x_0$, then either we obtain a monotonic sequence which leads us to a contradiction, or we obtain $x_{m-1} \geq x_m$ for some fixed m and we use the first case to obtain a contradiction. □

Open Problem 3.5. *Investigate the dynamics of Eq. (3.8).*

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